OPTIMAL CONTROL OF AN INVERTED PENDULUM USING A DIGITAL DEADBEAT RESPONSE PREDICTION OBSERVER

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Abstract: *This paper presents an optimal control algorithm of an inverted pendulum with Linear Quadratic Regulator in which a digital prediction observer with deadbeat response is used to estimate the unmeasured state variables. Optimal control is a powerful algorithm considers the limitations in state variables and system actuators. The main weak point of optimal control in a practical view is its complete dependence in sense or estimation of state variables. A digital prediction observer offers the fastest way of estimation of unmeasured state variables in a digital-based control system. The successful results of implementing of the proposed algorithm in this paper show the effectiveness of the algorithm..*

Keywords: Deadbeat response, Digital observer, Inverted Pendulum (IP), Optimal Control.

1 Introduction

The inverted pendulum is a classical control problem, which involves developing a system to balance a pendulum and belongs to the class of under-actuated mechanical systems having fewer control inputs than degrees of freedom (A detailed description of The Inverted Pendulum system is considered in [1]). This renders the control task more challenging making the inverted pendulum system a classical benchmark for testing different control techniques (Useful techniques related with the proposed algorithm in this paper could be found in [3,4,6,7,9]). There are a number of different versions of the IP system offering a variety of interesting control challenges. A single rod on a cart IP is a rod mounted on a moving cart that can rotate on its pivot as shown in figure 1. This system has tow equilibrium points that one of them is stable (fig. 1.b) and another one is unstable (fig. 1.a).

Stability and un-stability of these equilibrium points is simply provable by Liapunov stability theory with different methods as in [7,8,11]. With the rod exactly centered above the motionless cart, there are no sidelong resultant forces on the rod and it remains balanced as shown in Figure 1.1a. In principle it can stay this way indefinitely, but in practice it never does. Any disturbance that shifts the rod away from equilibrium, gives rise to forces that push the rod farther from this equilibrium point.

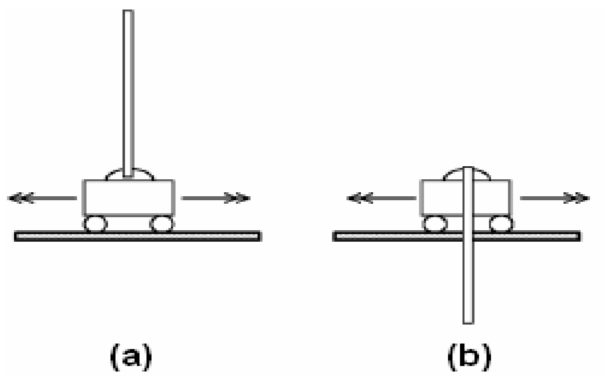

Fig. 1: Equilibrium points

Optimal control is a good choice of existing control algorithms for the systems in which state variables should be kept in complete restricted bounds and also actuators of the system are in danger of being saturated or damaged (design of an algorithm regardless of energy shaping of the system functions in nonlinear systems is frequently not useful, [7]-[9]). In order to implement the algorithm on a system, all state variables should be accessible simultaneously, therefore the estimation of some variables is vital in controlling the complex nonlinear variables.

Digital estimation is a software process of estimation of some state variables by means of the other state variables of the system ([10]). The best criterion in design of state observers is how fast the estimation error vector of the system tends to zero (a useful research in developing the observer can be found in [6]).

The described algorithm in this paper not only uses a deadbeat response estimation of the un-measurable state variables of the system, but also it is basically a prediction observer. In other words not only the error dynamics of the observed state variables tend to zero in the sampling periods less that the order of the system, but also because of the prediction feature of the designed estimation process, it is in one period ahead the supposed estimation time duration.

The paper is organized as follows: Section 2 presents a brief overview of the complete system and deals with the mathematical dynamical model of the system. Section 3 goes through the main steps in the design of the estimation algorithm and their digital implementation and other practical issues. Section 4 presents the method of design of the observer-based optimal controller and finally several conclusions are drawn in Section 5.

2 System Overview

Dynamical equation of an IP is presented in (1) where M is the mass of cart; *m* is the mass at the Centre Of Gravity (COG) of the pendulum; I is the moment of inertia of the pendulum; l is the distance from the COG of the pendulum to the pivot; *x* is the horizontal displacement of the cart; *g* is the gravitational acceleration; θ is the rod angular displacement and K and α are the coefficients derived from the practical experiments of the actual system (A detailed description of deriving (1) can be found in [12]).

These equations deduced from the linearization of nonlinear dynamical equations of the system around the unstable equilibrium point of the system and consequently are accurate only in the neighbourhood of the equilibrium point.

$$\begin{bmatrix}\dot\theta\\ \ddot\theta\\ \dot x\\ \ddot x\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ \frac{mgl(m+M)}{p} & 0 & 0 & \frac{-mlb}{p}\alpha(m+M)\\ 0 & 0 & 0 & 1\\ \frac{m^2gl^2}{p} & 0 & 0 & \frac{-(I+ml^2)b}{p}\alpha(m+M)\end{bmatrix}\begin{bmatrix}\theta\\ \dot\theta\\ x\\ \dot x\end{bmatrix}+\begin{bmatrix}0\\ \frac{ml}{p}K(m+M)\\ 0\\ \frac{I+ml^2}{p}K(m+M)\end{bmatrix}V$$

$$y=\begin{bmatrix}1&0&0&0\\0&0&1&0\end{bmatrix}\begin{bmatrix}\theta\\ \dot\theta\\ x\\ \dot x\end{bmatrix}+\begin{bmatrix}0\\0\end{bmatrix}u$$

$$p=I(M+m)+Mml^2 \qquad (1)$$

Implementation of a digital control algorithm on a microprocessor system is completely dependent on the sampling rate of the state variables. The smaller the period of sampling, the more accurate will be the estimation of the model of the system. In order to increase the sampling rate, a more complicated system should be designed which in many cases is not desirable or even viable. In any ways the sampling frequency should be tow times bigger that the Nyquist frequency of the system to prevent the folding phenomenon to happen.

In many systems with observing the minimum number of state variables, the sampling frequency of the system is small. So the digitalized model does not represent an accurate model of the system. This problem gets worse when a number of state variables of the system are un-measurable and should be estimated. In these systems the best solution to the estimation problem is to increase the dynamic of the estimation error of the state variables. In this case the values of the state variables are updated in every few sampling periods with the measurable state variables, such as done in using of the algorithm of digital prediction observer with deadbeat response.

3 Design of Digital Prediction Observer with Deadbeat Response

In practice in systems with measured and estimated state variables, reduced-order observers are used as presented in [10] and a hybrid algorithm is considered in [6]. In designing the digital prediction observer with deadbeat response for the designed Inverted Pendulum system, full-order observer is used. This is done because the

bandwidth of the system with full-order observer is less that the reduced-order one. So the system with full-order observer has a better high-frequency noise-rejection characteristic than the same system with reduced-order observer.
A linear model of the IP dynamics was adopted for the purpose of designing the optimal controller:

$$\begin{aligned} x(k+1) &= Gx(k) + Hu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \qquad (2)$$

The dynamic error equation of the observer can be formulated in form of (3).

$$e(k+1) = (G - K_e C)e(k) \qquad (3)$$

To simplify the procedure of design of the observer, the problem is changed to the dual form of designing the state feedback in (4):

$$\begin{aligned} \xi(k+1) &= G^* \xi(k) + C^* v(k) \\ v(k) &= -K_e^* \xi(k) \end{aligned} \qquad (4)$$

In which v(k) is the state feedback vector. In order to have a deadbeat response control system, the problem is changed into finding the feedback matrix K^*_e in (5).

$$\begin{bmatrix} \xi_1(k+1) \\ \xi_2(k+1) \\ \xi_3(k+1) \\ \xi_4(k+1) \end{bmatrix} = G^* \begin{bmatrix} \xi_1(k) \\ \xi_2(k) \\ \xi_3(k) \\ \xi_4(k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1(k) \\ v_2(k) \end{bmatrix}$$

$$v(k) = -K_e^* \xi(k) = -K\xi(k)$$
(5)

Where $K = K^*_e$.
Four linear independent vectors can be chosen as in (6) .In Inverted Pendulum system $n_1 = 2$, $n_2 = 2$ where n_is are Kronecker-invariant coefficients.

$$F = [H_1 \vdots GH_1 \vdots H_2 \vdots GH_2] \qquad (6)$$

Now let f_i be the η_ith row vector of F^{-1} where:

$$\begin{aligned} \eta_1 &= n_1 = 2 \\ \eta_2 &= n_1 + n_2 = 4 \end{aligned} \qquad (7)$$

By choosing the row vectors of f_1 and f_2 in terms of η_1 and η_2 rows of F^{-1}, (8) is deduced:

$$T = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}^{-1} \qquad (8)$$

Where:

$$S_1 = \begin{bmatrix} f_1 \\ f_1 G \end{bmatrix}, \quad S_2 = \begin{bmatrix} f_2 \\ f_2 G \end{bmatrix} \qquad (9)$$

With similarity transformation of the system by T, the state feedback gain matrix K is defined as (10).

$$\begin{aligned} K &= B\Delta T^{-1} \\ B &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \\ \Delta &= \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} & \delta_{14} \\ \delta_{21} & \delta_{22} & \delta_{23} & \delta_{24} \end{bmatrix}. \end{aligned} \qquad (10)$$

Finally with (11) the error vector shows a deadbeat response characteristic:

$$K_e^* = K \qquad (11)$$

The important note is that Δ is not unique. So in theory infinite Ks could be taken. But in practice optimal control method limits the number of choices.

4 Optimal Control Design

If the control process is lasting infinitely, the gain feedback matrix of K(k) is a constant matrix. (12) is the Performance Index of the system:

$$J = \frac{1}{2} \sum_{k=0}^{\infty} \left[x^*(k) Q x(k) + u^*(k) R u(k) \right] \qquad (12)$$

Where Q is a positive definite or semi-positive definite Hermitian matrix and R is a positive definite Hermitian matrix. To obtain a solution to minimize the performance index (12), the following Riccati equation is considered:

$$\begin{aligned} P &= Q + G^*(P^{-1} + HR^{-1}H^*)^{-1} G \\ &= Q + G^* PG - G^* PH(R + H^* PH)^{-1} H^* PG \end{aligned} \qquad (13)$$

where P is a Hermitian or a real positive definite symmetric matrix. Determining the weighting matrices and minimizing these cost equations, designs the controller ([10]). The underlying question here is how to determine the weighting matrices. Many methods have been devised, but one effective method is Bryson's Rule which weights each input and output as the inverse of the squared maximum desired value ([11]). After calculating Q and R, these two costs can be implemented and combined to form a controller known as the Linear Quadratic Gaussian (LQG) controller ([3]). Implementing this type of controller would be quite involved since it would be necessary to keep track of the state variables internally. These internal representations would be used to calculate the errors between the measurements and expected state variables. The error would then be minimized by the Kalman filter and the LQR control law would be applied using the estimated state variables. By calculating P from (13), the gain feedback matrix K is yielded with (14).

$$K = (H^* PH + R)^{-1}(H^* PG) \tag{14}$$

With this K, the following Quadratic performance index (15) is minimized.

$$J(u) = \sum_{n=1}^{\infty} (x[n]^T Qx[n] + u[n]^T Ru[n]) \tag{15}$$

5 Practical Results

The procedures of designing the controller and observer discussed in sections 3 and 4 were simulated and tested using the digitalized dynamic model of the system represented by (2). The sampling frequency of the system was deliberately reduced to 124 samples per second, where the Nyquist frequency of the actual designed IP system is 60 (with regard to nonlinearity characteristic of the system).
In designed IP system:

$$G^* = \begin{bmatrix} 1.000 & 0 & 0 & 0 \\ 0.061 & 0.932 & -0.002 & -0.085 \\ 0.001 & 0.054 & 1.026 & 0.8459 \\ 0.000 & 0.001 & 0.063 & 1.0266 \end{bmatrix}$$

And

$$\Delta = \begin{bmatrix} 0 & 0 & 1.017 & 529.8431 \\ 1 & 0 & 0 & 3.9857 \end{bmatrix}$$

So

$$K = B\Delta T^{-1} = \begin{bmatrix} 1.3 & 9.7 & 529.8 & 8547.4 \\ 0.3 & 16.5 & 4 & 64.9 \end{bmatrix}$$

With this K, the error vector of the digital prediction designed observer shows a deadbeat response characteristic.
In designed IP system,

$$Q^{\frac{1}{2}} = \begin{bmatrix} \frac{1}{(0.3491)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{(0.63)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

And

$$rank([(Q^{1/2})^* \vdots G^*(Q^{1/2})^* \vdots \ldots \vdots (G^*)^{n-1}(Q^{1/2})^*]) = 4$$

So in the basis of Liapunov stability criterion, the designed state feedback for the system is asymptotically stable ([11]).
The response to the designed feedback gain and prediction observer is completely satisfying as depicted in Figures 5.1 and 5.2.
Figures 5.3 and 5.4 depict a result of implementing the procedure on a designed IP system. As shown in the figures, although the sampling frequency is dangerously near the minimum limit of the possible frequency to prevent the folding phenomenon to happen and the success of control process is questionable, because of the prediction and deadbeat response characteristics of the designed observer, the IP is successfully controlled.

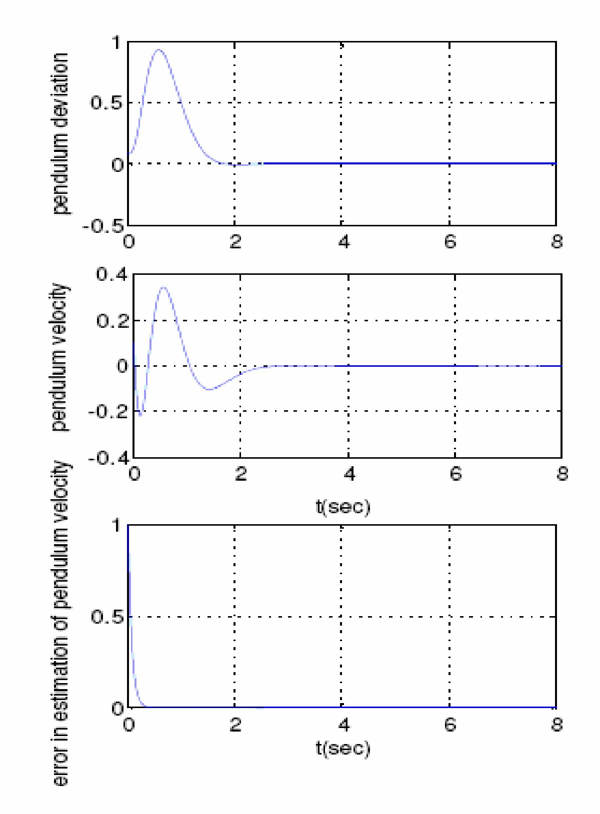

Fig 5.1: Desired pendulum behaviour of the IP system

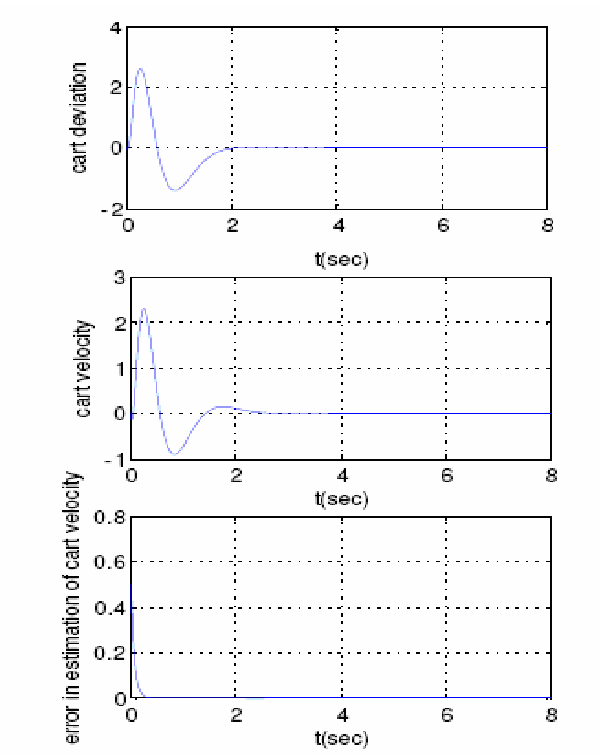

Fig 5.2: Desired cart behaviour of the IP system

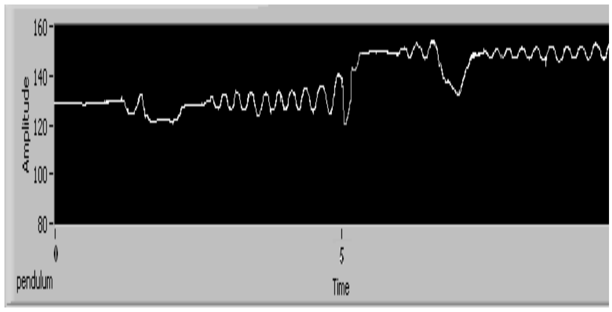

Fig 5.3: pendulum behaviour of designed IP system

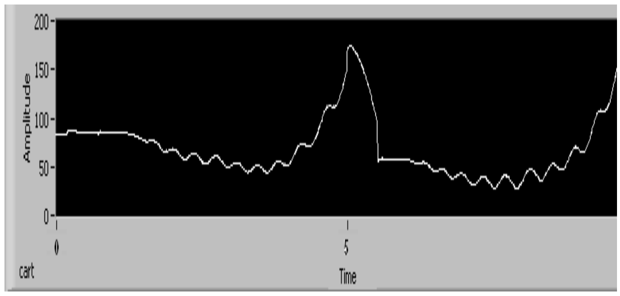

Fig 5.3: cart behaviour of designed IP system

6 Conclusion

This paper has presented an application approach which focuses on the real-time control of a horizontally driven inverted pendulum using optimal control approach and a digital prediction observer with deadbeat response. It has successfully been shown that for a class of under-actuated mechanical systems having fewer control inputs than degrees of freedom, like Inverted Pendulum, when sampling frequency of the measurable state variables is near the frequency in which folding phenomenon happens, combination of digital optimal control for controller design and a digital prediction observer with deadbeat response for observer design is applicable and effective. The proposed algorithm has been implemented on an actual Inverted Pendulum and the results show the effectiveness of the algorithm.

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