PID Type Iterative Learning Control with Optimal Variable Coefficients

Ali Madady
Tafresh University
Electrical Eng. Department

Masoud Shafiee
Amir-Kabir University of Technology
Electrical Eng. Department

drmadady@yahoo.com

Abstract: *Iterative learning control (ILC) is a technique to make use of the repetitiveness of the tasks a system is commanded to execute in a fixed finite time interval. In this paper, we assume that the system to be controlled is discrete time and described by linear state space equations. We present a PID type iterative learning control updating law with variable coefficients. An optimal procedure is used to determine the coefficients of PID type ILC so that the norm of error between the output of systems and desired trajectory is become minimize. The simulation example is included to illustrate the effectiveness of the proposed method.*

Keywords: Iterative Learning, PID Controller, Optimal design Control, Repetitive Processes.

1 Introduction

Mankind tries to extend the ability of learning to the engineering systems. One of these efforts is the designing and implementing of learning control systems. Important methods which are available in the learning control systems and at first were proposed in 1984 [1], are *Iterative Learning Control Systems* (ILCS).

Motivation of ILCS was where there are many industrial instances, that the system must do periodically a certain task. Examples of such systems are robot manipulators that are required to repeat a given task to high precision, chemical batch processes, or more generally, the class of tracking systems. If the conditions of system work are the same in all iterations, then the resultant errors will be equal in the all iterations. Thus by saving the input of system and resulted errors in iteration j, we can use of them to determine the system input for iteration $j+1$ in order to increase the performance and reducing the error. In recent decades this subject has been attended by researchers so much, and very useful developments have been happened in both theory and practical implementation. Nowadays this is a professional field in the control science. Interested readers for more studying about the subject can refer to [2-4].

The principle of ILCS is that, during the execution of control algorithm in the j th iteration, some data as errors are recorded. These are used by the learning algorithm in the execution $j+1$ for improving the control inputs and progressively reducing the output errors and increasing the performance of close loop system. Finally after a number of repeated trials, the system should obtain an appropriate control input, so that this input produces the desired output. Therefore the input of the controlled system can be written as follows:

$$u_{j+1}(i) = u_j(i) + \Delta u_{j+1}(i)$$

where $\Delta u_{j+1}(i)$ is a modifier term and denotes the change of system input in iteration $j+1$ relative to iteration j.

Difference between the presented methods in ILCS, is that how to determine $\Delta u_{j+1}(i)$, and determination of $\Delta u_{j+1}(i)$ shows the method of learning which has been used.

There are some efforts in [5-11] to determine $\Delta u_{j+1}(i)$ as a parametric optimization problem. In these studies, have been suggested specific control laws for assigning $\Delta u_{j+1}(i)$. In these control laws, there are optional parameters which have been determined to minimize a given cost function. In [12], we proposed a new P-type optimal method to determine $\Delta u_{j+1}(i)$. In this paper we extend this method to PID case.

The rest of the paper is organized as follows: Section 2 defines our problem. In section 3, this problem is solved. The convergence of the given ILC algorithm is analyzed in section 4. In section 5 we give an illustrative simulation example. Section 6 contains conclusion.

2 Problem Statement

Suppose the underlying single input single output (SISO) discrete-time repetitive system described by:

$$\begin{aligned} x_j(i+1) &= Ax_j(i) + Bu_j(i) \\ y_j(i) &= Cx_j(i) \\ i = 0,1,\ldots,M &,\ j = 0,1,\ldots \\ x_j(0) &= x_0 \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ denote the state , the input and the output respectively. Integer independent variables i and j respectively denote the time variable and the operation or iterations number. Integer M is the time duration of iterations. A , B and C are real-valued coefficients with appropriate dimensions. The x_0 is the systems initial condition, which is assumed be unknown.

We define the problem of this paper as follows: Suppose a desired output trajectory $y_d(i)$ is given. Utilizing the PID strategy, determine the control input sequence of system (1), such that with increasing the number of repetition the error between $y_j(i)$ and $y_d(i)$ become small as possible so that the following tracking can be established:

$$\lim_{j \to \infty}(y_d(i) - y_j(i)) = 0 \quad \text{for } i = 1,2,\ldots,M \tag{2}$$

3 Problem Solution

3.1 PID Type Iterative Learning Controller

We consider the following updating law to determine the input of system (1):

$$\begin{aligned} u_{j+1}(i) &= u_j(i) + \Delta u_{j+1}(i) \\ i = 0,1,\ldots,M-1 &,\ j = 0,1,\ldots \end{aligned} \tag{3}$$

where $\Delta u_{j+1}(i)$ is a modifier term.

Here, according to PID strategy $\Delta u_{j+1}(i)$ is chosen as follows:

$$\begin{aligned} \Delta u_{j+1}(i) = k_P(j+1)e_j(i+1) + k_I(j+1)\sum_{m=1}^{i+1} e_j(m) \\ + k_D(j+1)\big(e_j(i+1) - e_j(i)\big) \end{aligned} \tag{4}$$

where:

$$e_j(i) = y_d(i) - y_j(i) \tag{5}$$

and $k_P(j+1)$, $k_I(j+1)$ and $k_D(j+1)$ are real coefficients, we call them proportional , integration and derivative learning gains respectively. These learning gains are variable that is these are assumed depend on variable j and should be determined in a suitable and optimal manner.

The system (1) is casual, that is the error vector in iteration j which is defined below is independent of learning gains in iteration $j+1$:

$$e(j) = \begin{bmatrix} e_j(1) & e_j(2) & e_j(3) & \cdots & e_j(M) \end{bmatrix}^T \tag{6}$$

(T denotes the Transpose)

Here we use this fact for determination the learning gains as optimal. That is we assume $e(j)$ is known and we determine the learning gains in iteration $j+1$ so that the following performance index is minimized:

$$\begin{aligned} J(j+1) = \|e(j+1)\|^2 + \lambda_P k_P^2(j+1) \\ + \lambda_I k_I^2(j+1) + \lambda_D k_D^2(j+1) \end{aligned} \tag{7}$$

where:

$$\|e(j+1)\|^2 = e^T(j+1)e(j+1)$$

and λ_P , λ_I , λ_D are positive weighting parameters introduced to limit the values of k_P ,k_I and k_D.

Physical interpretation of the given cost function is that we wish to close the system output to desired output trajectory without many large learning gains.

3.2 Dynamic of the Error Vector

We try to obtain the dynamic of the error vector $e(j)$ .For this purpose from (1) we get:

$$x_j(i) = A^i x_0 + \sum_{k=0}^{i-1} A^{i-1-k} Bu_j(k) \tag{8}$$

It is easy to obtain the following relation from (1) and (8):

$$Y(j) = G_P U(j) + G_0 x_0 \qquad (9)$$

where:

$$Y(j) = \begin{bmatrix} y(1,j) \\ y(2,j) \\ y(3,j) \\ \vdots \\ y(M,j) \end{bmatrix}, \quad U(j) = \begin{bmatrix} u(0,j) \\ u(1,j) \\ u(2,j) \\ \vdots \\ u(M-1,j) \end{bmatrix} \qquad (10)$$

$$G_P = \begin{bmatrix} CB & 0 & 0 & 0 & \cdots & 0 \\ CAB & CB & 0 & 0 & \cdots & 0 \\ CA^2B & CAB & CB & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ CA^{M-2}B & CA^{M-3}B & \cdots & CAB & CB & 0 \\ CA^{M-1}B & CA^{M-2}B & \cdots & CA^2B & CAB & CB \end{bmatrix} \qquad (11)$$

Using relation (9) we can write:

$$Y(j+1) - Y(j) = G_P U(j+1) + G_0 x_0 - G_P U(j) - G_0 x_0$$

or:

$$Y(j+1) = Y(j) + G_P V(j) \qquad j = 0,1,\dots \qquad (12)$$

where:

$$V(j) = U(j+1) - U(j) \qquad (13)$$

From (12) we have:

$$Y_d - Y(j+1) = Y_d - Y(j) - G_P V(j) \qquad (14)$$

where:

$$Y_d = \begin{bmatrix} y_d(1) & y_d(2) & \cdots & y_d(M) \end{bmatrix}^T \qquad (15)$$

Considering the definition of the vectors $e(j)$, $Y(j)$ and Y_d, which are given respectively by (6), (10) and (15), we get:

$$e(j) = Y_d - Y(j)$$

Therefore equation (14) will become as follows:

$$e(j+1) = e(j) - G_P V(j) \qquad j = 0,1,\dots \qquad (16)$$

The above equation is the dynamic of error vector $e(j)$.

3.3 Optimal Determination of Learning Gains

Utilizing (3) and (4) we get:

$$\begin{aligned} u_{j+1}(i) = u_j(i) &+ k_P(j+1)e_j(i+1) \\ &+ k_I(j+1)\sum_{m=1}^{M} e_j(m) \\ &+ k_D(j+1)\big(e_j(i+1) - e_j(i)\big) \end{aligned} \qquad (17)$$

$$i = 0,1,\dots,M-1$$

Considering the definition of the vectors $e(j)$ and $Y(j)$, we can rewrite the relations (17) as following compact form:

$$\begin{aligned} V(j) = \{&k_P(j+1)I + k_I(j+1)H_I \\ &+ k_D(j+1)H_D\}e(j) \end{aligned} \qquad (18)$$

where I is identity matrix, H_I and H_D are as follows:

$$H_I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$$

$$H_D = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & 1 & 0 & 0 \\ 0 & 0 & & -1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \qquad (19)$$

Substituting for $V(j)$ from (18) into (16) yields:

$$e(j+1) = G_c e(j) \qquad j = 0,1,\dots \qquad (20)$$

where:

$$\begin{aligned} G_c = I - k_P(j+1)G_P &- k_I(j+1)G_I \\ &- k_D(j+1)G_D \end{aligned} \qquad (21)$$

and:

$$G_I = G_P H_I \quad , \quad G_D = G_P H_D \qquad (22)$$

The equation (20) is the dynamic of closed loop system.

From (20) and (21) it is easy to show:

$$J(j+1)=e^T(j)e(j)+K^T(j+1)\{\Lambda+\Psi(j)\}K(j+1)-2\Phi^T(j)K(j+1) \quad (23)$$

where:

$$K(j+1)=\begin{bmatrix}k_P(j+1) & k_I(j+1) & k_D(j+1)\end{bmatrix}^T \quad (24)$$

$$\Lambda=\begin{bmatrix}\lambda_P & 0 & 0\\ 0 & \lambda_I & 0\\ 0 & 0 & \lambda_D\end{bmatrix} \quad (25)$$

$$\Psi(j)=\begin{bmatrix}e^T(j)G_P^TG_Pe(j) & e^T(j)G_P^TG_Ie(j) & e^T(j)G_P^TG_De(j)\\ e^T(j)G_I^TG_Pe(j) & e^T(j)G_I^TG_Ie(j) & e^T(j)G_I^TG_De(j)\\ e^T(j)G_D^TG_Pe(j) & e^T(j)G_D^TG_Ie(j) & e^T(j)G_D^TG_De(j)\end{bmatrix} \quad (26)$$

$$\Phi^T(j)=\begin{bmatrix}e^T(j)G_Pe(j) & e^T(j)G_Ie(j) & e^T(j)G_De(j)\end{bmatrix} \quad (27)$$

The gradient (derivative) $J(j+1)$ respect to $K(j+1)$ is simply:

$$\frac{\nabla J(j+1)}{\nabla K(j+1)}=2\{\Lambda+\Psi(j)\}K(j+1)-2\Phi(j) \quad (28)$$

It is easy to show that the following decomposition for $\Psi(j)$:

$$\Psi(j)=E^T(j)G^TGE(j) \quad (29)$$

where:

$$E(j)=\begin{bmatrix}e(j) & 0 & 0\\ 0 & e(j) & 0\\ 0 & 0 & e(j)\end{bmatrix},\ G=\begin{bmatrix}G_P & G_I & G_D\end{bmatrix} \quad (30)$$

From (25) and (29) we conclude that the symmetric matrix $\{\Lambda+\Psi(j)\}$ is positive definite and hence it is invertible, and also its inverse is symmetric positive definite. Thus, the equation $\frac{\nabla J(j+1)}{\nabla K(j+1)}=0$ has solution for $K(j+1)$.

Solving the equation $\frac{\nabla J(j+1)}{\nabla K(j+1)}=0$ yields to optimum value for $K(j+1)$:

$$K(j+1)=\{\Lambda+\Psi(j)\}^{-1}\Phi(j) \quad (31)$$

Using (28), we get:

$$\frac{\nabla^2 J(j+1)}{\nabla K^2(j+1)}=2\{\Lambda+\Psi(j)\} \quad (32)$$

The symmetric matrix $\{\Lambda+\Psi(j)\}$ is positive definite, hence $K(j+1)$ causes the performance index $J(j+1)$ be global minimum.

4 Convergence Analysis

The ILC algorithm obtained in the previous section has several useful properties as can be seen as follows.

Theorem 1- For the ILC algorithm defined by equations (4) and (31):

(a) The performance index satisfies the interlacing monotonically condition:

$$\|e(j)\|^2 \geq J(j+1) \geq \|e(j+1)\|^2 \quad (33)$$

with equality holding if and only if:

$$k_P(j+1)=k_I(j+1)=k_D(j+1)=0$$

(b) The proportional, integration and derivative learning gains satisfy the condition:

$$\sum_{j=0}^{\infty}\{\lambda_Pk_P^2(j+1)+\lambda_Ik_I^2(j+1)+\lambda_Dk_D^2(j+1)\}<\infty \quad (34)$$

and hence:

(c) $$\lim_{j\to\infty}k_P(j)=0,\ \lim_{j\to\infty}k_I(j)=0,\ \lim_{j\to\infty}k_D(j)=0 \quad (35)$$

Note- (a) states that the algorithm is a decent algorithm as the norm of the error is monotonically non-increasing in j, and the 'energy costs' from the first to the last trial are bounded, whilst (b) and (c) indicates that the learning rate becomes slower as the algorithm progresses to convergence (see Theorem 2).

Proof:

(a) Consider (7), since λ_P , λ_I and λ_D are positive coefficients we have:

$$J(j+1) \geq \|e(j+1)\|^2$$

with equality holding if and only if:

$$k_P(j+1)=k_I(j+1)=k_D(j+1)=0$$

Substituting for $K(j+1)$ from (31) into (23) yields:

$$J(j+1)=\|e(j)\|^2-\Phi^T(j)\{\Lambda+\Psi(j)\}^{-1}\Phi(j)$$

The matrix $\{\Lambda+\Psi(j)\}^{-1}$ is positive definite, hence if $\Phi(j)\neq 0$ or equivalently $K(j+1)\neq 0$, we conclude:

$$J(j+1)<\|e(j)\|^2$$

But, if $K(j+1)=0$ we get:

$$J(j+1)=\|e(j)\|^2$$

(b) From (7) and (33), we have:

$$\|e(j+1)\|^2+\lambda_P k_P^2(j+1)+\lambda_I k_I^2(j+1) + \lambda_D k_D^2(j+1)\leq\|e(j)\|^2$$

Apply the induction to the above relation, the following inequality is resulted for all j:

$$\|e(j+1)\|^2+\sum_{l=0}^{j}\left\{\lambda_P k_P^2(l+1)+\lambda_I k_I^2(l+1) + \lambda_D k_D^2(l+1)\right\}\leq\|e(0)\|^2$$

which proves (b) and hence (c).

■

Theorem 2- If matrix G_P which is defined in (11) is positive in the sense that $G_P+G_P^T$ is positive definite, then the following iterative learning convergence condition is obtained:

$$\lim_{j\to\infty}\|e(j)\|=0 \tag{36}$$

Proof:
From (35) and (31) the following result is obtained:

$$\lim_{j\to\infty}\Phi(j)=0 \tag{37}$$

from (37) and (27) we conclude:

$$\lim_{j\to\infty}e^T(j)G_P e(j)=0 \tag{38}$$

or:

$$\lim_{j\to\infty}e^T(j)\left(G_P+G_P^T\right)e(j)=0 \tag{39}$$

It is clear from the assumption that $G_P+G_P^T$ is positive definite, is resulted there exists a real number ρ^2 such that:

$$e^T(j)\left(G_P+G_P^T\right)e(j)\geq\rho^2 e^T(j)e(j)\quad\forall e(j) \tag{40}$$

from (39) and (40) we get (36).

■

In summary, by using the updating rule (4) and the performance index (7), the positivity condition on G_P ensures that:

(a) The iterative learning control tracking error sequence $\{e(j): j=0,1,2,...\}$ converges in norm to zero, i.e. the iterative learning control algorithm has guaranteed convergence of learning.

(b) This convergence has the important property that the error norm sequence is monotonic.

5 Simulation Results

To demonstrate the effectiveness of the new parameter optimization based ILC algorithm which is defined by (4) and (31), consider a plant having the following model:

$$x_j(i+1)=\begin{bmatrix}0.1 & 0 & 0.25\\ 2 & -0.5 & -0.2\\ -1 & 1 & 0.4\end{bmatrix}x_j(i)+\begin{bmatrix}0.5\\0\\-1\end{bmatrix}u_j(i)$$
$$y_j(i)=\begin{bmatrix}1 & 0 & 0\end{bmatrix}x_j(i)$$
$$i=0\leq i\leq 100\ ,\ j=0,1,...$$

For this plant, MATLAB indicates that the eigenvalues of $G_P+G_P^T$ lie between 0.4054 and 2.7353 and hence G_P is a positive matrix.

The desired output trajectory is chosen as follows:

$$y_d(i)=0.1i\sin(i\tfrac{\pi}{25})\quad 0<i\leq 100$$

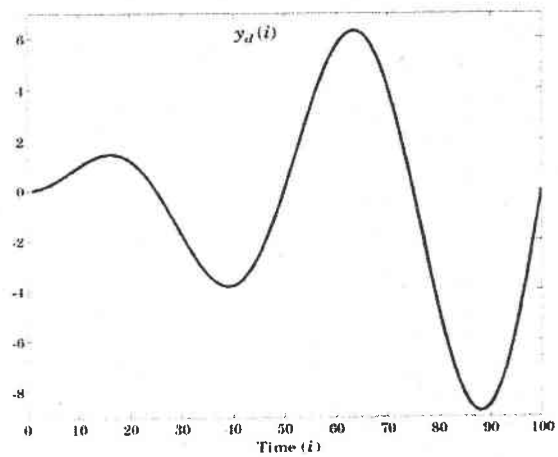

Figure 1: Desired output trajectory $y_d(i)$

In order to select a value for λ_P, λ_I and λ_D from performance index (7), the selection $\lambda_P=\lambda_I=\lambda_D=0.01$ is chosen to provide numerical solutions of the evaluations of $k_P(j+1), k_I(j+1)$ and $k_D(j+1)$ at very small error values. The results are shown in Figs. 2 - 3.

The results confirm the theoretical prediction that $\|e(j)\|$ and $k_P(j), k_I(j), k_D(j)$ converge to zero as $j \to \infty$ and that the convergence of the error norm is monotonic. This is due to Theorems 1 and 2, which state that the positivity of the plant is a sufficient condition for monotonic convergence to zero.

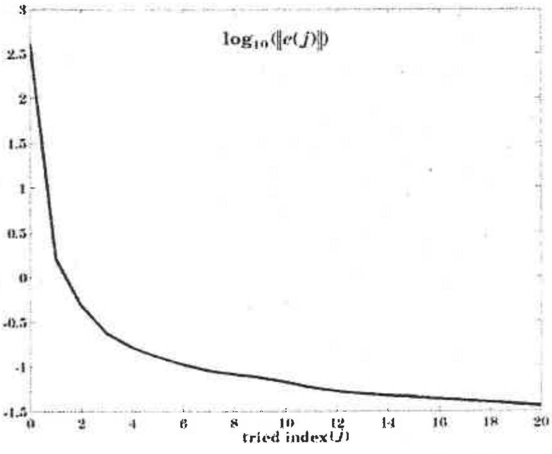

Figure 2: Norm of error in log 10 ($\log_{10}\|e(j)\|$)

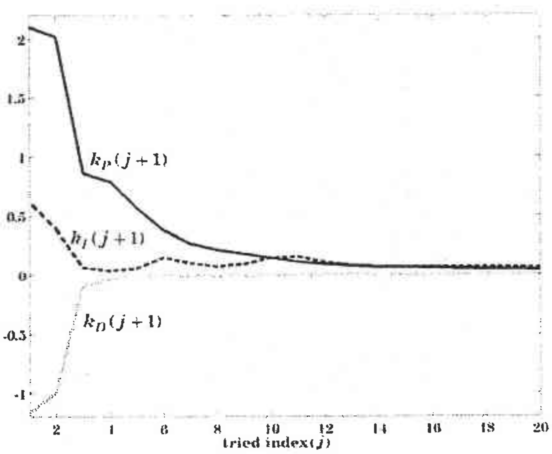

Figure 3: Value of parameters $k_P(j+1), k_I(j+1)$ and $k_D(j+1)$

6 Conclusions

In this paper, parameter optimization based iterative learning control was introduced as a new paradigm to solve the ILC problem when the original plant is a discrete-time LTI system. The resulting algorithm is PID type and has guaranteed monotonic convergence to zero if the original system satisfies a positivity condition. Because of its computational simplicity, this new ILC algorithm is potentially straightforward to implement in real-time applications. The weighting parameters in the chosen performance index influence convergence rates in a natural manner and intuitively add a degree of robustness to the methodology.

The effectiveness of the proposed PID type ILC is illustrated by simulation results. However, formulating and solving the problem when the underlying repetitive system is nonlinear merit further researches.

References

[1] S. Arimoto, S. Kawamura and F. Miyazaki, "Bettering operation of robots by learning," *Journal of Robotic Systems*, vol. 1, pp. 123 140, 1984.

[2] J. X. Xu and Y. Tan, *Linear and Nonlinear Iterative Learning Control.* New York: Springer-Verlag, 2003.

[3] A. Madady, "Self-tuning iterative learning control for time variant systems," in *Proceedings of the 44th IEEE Conference on Decision and Control*, Seville, Spain, December 12 15, 2005.

[4] A. Madady, "A model reference adaptive iterative learning control system," in *Proceedings of the 32nd Annual Conference of the IEEE Industrial Electronics Society*, Paris – *France*, November 7 – 10, 2006.

[5] J. –X. Xu and Y. Tan, "Robust optimal design and convergence properties analysis of iterative learning control approaches," *Automatica*, vol. 38, pp. 1867 1880, 2002.

[6] D. H. Owens, and K. Feng, "Parameter optimization in iterative learning control," *International Journal of Control*, vol. 76, no. 11, pp. 1059 1069, 2003.

[7] J. Hatonen and D. H. Owens, "Convex modifications to an iterative learning control law," *Automatica*, vol. 40, pp. 1213 1220, 2004.

[8] D. H. Owens and J. Hatonen, "Iterative learning control — an optimization paradigm," *Annual Reviews in Control*, vol. 29, pp. 57 70, 2005.

[9] S. S. Saab, "Optimal selection of the forgetting matrix into an iterative learning control algorithm," *IEEE Transactions on Automatic Control*, vol. 50, no. 12, December, pp. 2039 2043, 2005.

[10] J. Hatonen, D. H. Owens and K. Feng, "Basis functions and parameter optimization in high-order iterative learning control," *Automatica*, vol. 42, pp. 287 – 294, 2006.

[11] A. Madady and Moham Reza Nasiri, "An optimal iterative learning control for continuous -time systems," in *Proceedings of the 32nd Annual Conference of the IEEE Industrial Electronics Society*, Paris – *France*, November 7 – 10, 2006.

[12] A. Madady and M. Shafiee, "P type iterative learning control with optimal gain," in *Proceedings of the 14th Iranian Conference of Electrical Engineering (ICEE2006)*, AmirKabir University of Technology – Tehran, May 16 – 18, 2006.