Stabilization of a class of Linear Time-Varying Systems with Time Delay

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Abstract: *In this paper, both stability and stabilization of a class of linear time-varying systems with multiple time delays are considered. A new controller using state feedback for stabilizing of this class of systems is designed. A numerical example is provided to verify the results established.*

Keywords: Stabilization, linear time-delay system.

1. Introduction

The important attention has been devoted to the stability and stabilization of time-invariant linear and nonlinear time delay systems as reported by[1]. Some research has been conducted towards the stability and stabilization of linear time-varying system with time delay in the state[1-3]; but in those papers, the states delayed dynamic which may have unbounded parameters were considered. In this paper, we consider the system output as well as the system states to have the unbounded parameters. Obviously this case is not a trivial case of the work which was done in[1-3].Thus the output would be of the form $y(t)=C(t)x(t)$,which $C(t)$ is considered to be unbounded, which implies that the system may be unstable, even if all the system's states are stable. This paper investigates both stability and stabilization independent of the delays of a class of linear time-varying delay systems. A new controller, since the unbounded outputs are considered using state feedback for stabilizing of such a class of systems is designed.. In section 2, the stability and stabilization of the plant under a point delay is considered. In section 3, the method is extended to cover the multi-delay systems. Eventually, a numerical example is provided in order to verify the theoretical results.

2. Plant model under a point delay

Consider the following linear time-varying system:

$$\dot{x}(t)=A_0(t)x(t)+A_1(t)x(t-h)+B(t)u(t) \qquad (1-a)$$

$$y(t)=C(t)x(t) \qquad (1-b)$$

$$x(t)=\varphi(t) \;\; ,-h\le t\le 0$$

where $x(t)\in\Re^n$ is the state, $u(t)\in\Re^r$ is the control input, $y(t)\in\Re^m$ is the output. $A_0(t), A_1(t), B(t), C(t)$ are continuous matrices in the interval $[0,\infty)$ with appropriate dimensions. $C(t)$ is considered to be differentiable. The initial value $\varphi(t)$ is continuous function in the interval $[-h,0]$.

2-1. Stability

Consider the system:

$$\dot{x}(t)=A_0(t)x(t)+A_1(t)x(t-h) \qquad (2-a)$$

$$y(t)=C(t)x(t) \qquad (2-b)$$

$$x(t)=\varphi(t) \;,-h\le t\le 0$$

where $x(t)\in\Re^n$ is the state, $y(t)\in\Re^m$ is the output. $A_0(t), A_1(t), C(t)$ are continuous matrices with appropriate dimensions. $C(t)$ is considered to be differentiable. The initial value $\varphi(t)$ is continuous function in the interval $[-h,0]$.

Theorem 1: System(2) is stable if there are positive definite symmetric matrices $P_{n\times n}(t), Q_{m\times m}(t)$ and positive definite matrix $K_{n\times n}(t)$ such that one of the following two conditions holds:

Condition1:

$$\dot{P}(t)+A_0^T(t)P(t)+P(t)A_0(t)+P(t)A_1(t)A_1^T(t)P(t)+ \dot{C}^T(t)Q(t)C(t)+C^T(t)\dot{Q}(t)C(t)+A_0^T(t)C^T(t)Q(t)C(t)+ C^T(t)Q(t)\dot{C}(t)+C^T(t)Q(t)C(t)A_0(t)+ C^T(t)Q(t)C(t)A_1(t)A_1^T(t)C^T(t)Q(t)C(t)+2I+K(t)=0$$

Condition2:

$$\begin{bmatrix} \dot{P}(t)+A_0^T(t)P(t)+ P(t)A_0(t)+\dot{C}^T(t)Q(t)C(t)+ A_0^T(t)C^T(t)Q(t)C(t)+ C^T(t)\dot{Q}(t)C(t)+ C^T(t)Q(t)\dot{C}(t)+ C^T(t)Q(t)C(t)A_0(t)+2I & P(t)A_1(t)+ C^T(t)Q(t)C(t)A_1(t) \\ A_1^T(t)P(t)+ A_1^T(t)C^T(t)Q(t)C(t) & -2I \end{bmatrix} < 0$$

Remark 2.1: Finite Difference method can be used to solve the condition1 numerically as well as it can be seen in [4].

Proof: Consider the following Lyapunov-Krasovskii candidate functional:

$$V(t)=x^T(t)P(t)x(t)+y^T(t)Q(t)y(t)+2\int_{t-h}^{t}\|x(\tau)\|^2 d\tau$$

Taking the derivative of $V(t)$ along the trajectory (2) and using $\dot{y}=\dot{C}x+C\dot{x}$,we get

$x=x(t), P=P(t), Q=Q(t), y=y(t), C=C(t)$

$$\dot{V}=(A_0x+A_1x(t-h))^T Px+x^T\dot{P}x+x^T P(A_0x+A_1x(t-h))+ (\dot{C}x+C(A_0x+A_1x(t-h)))^T QCx+x^T C^T\dot{Q}Cx+ x^TC^TQ(\dot{C}x+C(A_0x+A_1x(t-h)))+2\|x\|^2-2\|x(t-h)\|^2$$

$$\dot{V}=x^TA_0^TPx+x^T(t-h)A_1^TPx+x^T\dot{P}x+ x^TPA_0x+x^TPA_1x(t-h)+x^T\dot{C}^TQCx+ x^TA_0^TC^TQCx+x^T(t-h)A_1^TC^TQCx+ x^TC^T\dot{Q}Cx+x^TC^TQ\dot{C}x+x^TC^TQCA_0x+ x^TC^TQCA_1x(t-h)+2\|x\|^2-2\|x(t-h)\|^2 \quad (3)$$

Lemma: For any vectors $\nu_1,\nu_2\in\Re^n$ and any positive definite matrix $M\in\Re^{n\times n}$,the following inequality holds[5]:

$$2\,\nu_1\nu_2\le\nu_1^T M\nu_2+\nu_2^T M^{-1}\nu_1$$

Using the lemma, the equation(3) becomes

$$\dot{V}(t)\le x^T(t)(\dot{P}(t)+A_0^T(t)P(t)+P(t)A_0(t)+ P(t)A_1(t)A_1^T(t)P(t)+\dot{C}^T(t)Q(t)C(t)+ C^T(t)\dot{Q}(t)C(t)+A_0^T(t)C^T(t)Q(t)C(t) +C^T(t)Q(t)\dot{C}(t)+C^T(t)Q(t)C(t)A_0(t)+ C^T(t)Q(t)C(t)A_1(t)A_1^T(t)C^T(t)Q(t)C(t)+2I)x(t)$$

If the condition1 holds, we would have

$\dot{V}\le -x^T(t)K(t)x(t)<0$

which implies that the system(2) is asymptotically stable.

The equality (3) can be rewritten in the following form:

$$\dot{V}(t)=F^T(t)\begin{bmatrix} A_0^T(t)P(t)+\dot{P}(t)+ P(t)A_0(t)+\dot{C}^T(t)Q(t)C(t)+ A_0^T(t)C^T(t)Q(t)C(t) +C^T(t)\dot{Q}(t)C(t)+ C^T(t)Q(t)\dot{C}(t) +C^T(t)Q(t)C(t)A_0(t)+2I & P(t)A_1(t)+ C^T(t)Q(t)C(t)A_1(t) \\ A_1^T(t)P(t)+ A_1^T(t)C^T(t)Q(t)C(t) & -2I \end{bmatrix}F(t)$$

where $F(t)=\begin{bmatrix} x(t) \\ x(t-h)\end{bmatrix}$. If the condition2 holds, we would have $\dot{V}(t)<0$,which implies system(2) is asymptotically stable.

Corollary 1: Consider the following linear time-invariant system with time delay:

$$\dot{x}(t)=A_0x(t)+A_1x(t-h) \quad (4-a)$$

$$y(t)=Cx(t) \quad (4-b)$$

$x(t)=\varphi(t)\ ,-h\le t\le 0$

The system(4) is stable if there are positive definite symmetric matrices $P_{n\times n}, Q_{m\times m}$ and a positive definite matrix $K_{n\times n}$ such that one of the following two conditions holds:

Condition1:

$$A_0^TP+PA_0+PA_1A_1^TP+A_0^TC^TQC+C^TQCA_0+ C^TQCA_1A_1^TC^TQC+2I+K=0$$

Condition 2:

$$\begin{bmatrix} A_0^T P + PA_0 + A_0^T C^T QC + C^T QCA_0 + 2I & PA_1 + C^T QCA_1 \\ A_1^T P + A_1^T C^T QC & -2I \end{bmatrix} < 0$$

2-2. Stabilization

Theorem2: Consider the system

$$\dot{x}(t) = A_0(t)x(t) + \sum_{i=1}^{m} A_i(t)x(t-h_i) + B(t)u(t) \quad (5)$$

$$x(t) = \varphi(t)\ ,\ \bar{h} \le t \le 0$$

$$\bar{h} = \max\{h_i\}, h_i > 0\ ,1 \le i \le L$$

where $x(t) \in \Re^n$, $u(t) \in \Re^r$, $A_i(t), i = 0,1,2,...,m$, $B(t)$ are the matrix functions of the delayed dynamic of appropriate dimensions, the initial vector $\varphi(t)$ is a continuous function given in the interval $[-h, 0]$. h is the time delay.

Assume that the matrix functions $A_i(t), i = 1,2,...,m$ are differentiable in $t \in \Re^+$. Assume that

$$rank\left[M_0(t_0), M_1(t_0), ..., M_{n-1}(t_0),\right] = n$$

where

$$M_0(t) = \left[B(t), A_{1,\alpha}(t), ..., A_{m,\alpha}(t)\right]\ ,$$

$$A_{i,\alpha}(t) = e^{\alpha h_i} A_i(t)$$

$$M_k(t) = -A_{0,\alpha}(t)M_{k-1}(t) + \frac{d}{dt}M_{k-1}(t)\ ,$$

$$k = 1,...,n-1\ ,\ A_{0,\alpha}(t) = A_0(t) + \alpha I$$

Then the system(5) is $\alpha -$stabilizable[2].

The following assumption is made on system(1):

Assumption1: System(1) is stabilizable.

Theorem3: The system(1) with condition (**A1**) and the controller

$$u(t) = -\left(B^T(t)P(t) + B^T(t)C^T(t)Q(t)C(t)\right)x(t)$$

is stable if there are positive definite symmetric matrices $P_{n\times n}(t), Q_{m\times m}(t)$ and a positive definite matrix $K_{n\times n}(t)$ such that one of the following two conditions holds:

(*Condition* 1'):

$$\begin{aligned}
&\dot{P}(t) + A_0^T(t)P(t) + P(t)A_0(t) + P(t)A_1(t)A_1^T(t)P(t) \\
&- 2P(t)B(t)B^T(t)P(t) - 2P(t)B(t)B^T(t)C^T(t)Q(t)C(t) \\
&- 2C^T(t)Q(t)C(t)B(t)B^T(t)P(t) + \dot{C}^T(t)Q(t)C(t) + \\
&C^T(t)\dot{Q}(t)C(t) + A_0^T(t)C^T(t)Q(t)C(t) + C^T(t)Q(t)\dot{C}(t) \\
&- 2C^T(t)Q(t)C(t)B(t)B^T(t)C^T(t)Q(t)C(t) \\
&+ C^T(t)Q(t)C(t)A_0(t) + C^T(t)Q(t)C(t)A_1(t)A_1^T(t)C^T(t)Q(t)C(t) \\
&+ 2I + K(t) = 0
\end{aligned}$$

(*Condition* 2'):

$$\begin{bmatrix} \begin{array}{l} A_0^T(t)P(t) - 2P(t)B(t)B^T(t)P(t) \\ - 2C^T(t)Q(t)C(t)B(t)B^T(t)P(t) \\ + \dot{P}(t) + P(t)A_0(t) \\ - 2P(t)B(t)B^T(t)C^T(t)Q(t)C(t) \\ + \dot{C}^T(t)Q(t)C(t) + A_0^T(t)C^T(t)Q(t)C(t) \\ - 2C^T(t)Q(t)C(t)B(t)B^T(t)C^T(t)Q(t)C(t) \\ + C^T(t)\dot{Q}(t)C(t) + C^T(t)Q(t)\dot{C}(t) \\ + C^T(t)Q(t)C(t)A_0(t) + 2I \end{array} & \begin{array}{l} P(t)A_1(t) + \\ C^T(t)Q(t)C(t)A_1(t) \end{array} \\ \begin{array}{l} A_1^T(t)P(t) + \\ A_1^T(t)C^T(t)Q(t)C(t) \end{array} & -2I \end{bmatrix} < 0$$

Proof: Consider the following Lyapunov-Krasovskii candidate functional:

$$V(t) = x^T(t)P(t)x(t) + y^T(t)Q(t)y(t) + 2\int_{t-h}^{t} \|x(\tau)\|^2 d\tau$$

Taking the derivative of $V(t)$ along the trajectory (1) and using $\dot{y} = \dot{C}x + C\dot{x}$, we get

$$x = x(t), P = P(t), Q = Q(t), y = y(t), C = C(t)$$

$$\begin{aligned}
\dot{V} = &\left(A_0x + A_1x(t-h) + Bu\right)^T Px + x^T\dot{P}x + \\
&x^T P\left(A_0x + A_1x(t-h) + Bu\right) + \\
&\left(\dot{C}x + C\left(A_0x + A_1x(t-h) + Bu\right)\right)^T QCx + \\
&x^TC^T\dot{Q}Cx + x^TC^TQ\left(\dot{C}x + C\left(A_0x + A_1x(t-h) + Bu\right)\right) \\
&+ 2\|x\|^2 - 2\|x(t-h)\|^2
\end{aligned}$$

Using $u = -\left(B^TP + B^TC^TQC\right)x$, we get

$$
\begin{aligned}
&\dot{V} = \left(A_0x + A_1x(t-h) - B\left(B^TP + B^TC^TQC\right)x\right)^T Px + x^T\dot{P}x + \\
&x^TP\left(A_0x + A_1x(t-h) - B\left(B^TP + B^TC^TQC\right)x\right) + \\
&\left(\dot{C}x + C\left(A_0x + A_1x(t-h) - B\left(B^TP + B^TC^TQC\right)x\right)\right)^T QCx \\
&+ x^TC^T\dot{Q}Cx + x^TC^TQ(\dot{C}x + C(A_0x + A_1x(t-h) - \\
&B(B^TP + B^TC^TQC)x)) + 2\|x\|^2 - 2\|x(t-h)\|^2 = \\
&\dot{V} = x^TA_0^TPx + x^T(t-h)A_1^TPx - x^TPBB^TPx - x^TC^TQCBB^TPx + \\
&x^T\dot{P}x + x^TPA_0x + x^TPA_1x(t-h) - x^TPBB^TPx \\
&- x^TPBB^TC^TQCx + x^T\dot{C}^TQCx + x^TA_0^TC^TQCx + \\
&x^T(t-h)A_1^TC^TQCx - x^TPBB^TC^TQCx - \\
&x^TC^TQCBB^TC^TQCx + x^TC^T\dot{Q}Cx + x^TC^TQ\dot{C}x + \\
&x^TC^TQCA_0x + x^TC^TQCA_1x(t-h) - x^TC^TQBB^TPx \\
&- x^TC^TQBB^TC^TQCx + 2\|x\|^2 - 2\|x(t-h)\|^2
\end{aligned} \tag{6}
$$

Using the lemma, the equality (6) becomes

$$
\begin{aligned}
&\dot{V} \le x^T(\dot{P}(t) + A_0^T(t)P(t) + P(t)A_0(t) + \\
&P(t)A_1(t)A_1^T(t)P(t) - 2P(t)B(t)B^T(t)P(t) \\
&- 2P(t)B(t)B^T(t)C^T(t)Q(t)C(t) \\
&- 2C^T(t)Q(t)C(t)B(t)B^T(t)P(t) + \\
&\dot{C}^T(t)Q(t)C(t) + C^T(t)\dot{Q}(t)C(t) + \\
&A_0^T(t)C^T(t)Q(t)C(t) + C^T(t)Q(t)\dot{C}(t) \\
&- 2C^T(t)Q(t)C(t)B(t)B^T(t)C^T(t)Q(t)C(t) \\
&+ C^T(t)Q(t)C(t)A_0(t) + \\
&C^T(t)Q(t)C(t)A_1(t)A_1^T(t)C^T(t)Q(t)C(t) + 2I)x
\end{aligned} \tag{7}
$$

If the condition $1'$ holds, the equality(7) turns to be

$$\dot{V} \le -x^T(t)K(t)x(t) < 0$$

This implies system(1) with controller $u = -\left(B^TP + B^TC^TQC\right)x$ is asymptotically stable.

The equality (6) can be rewritten as follows:

$$
\dot{V}(t) = z^T(t)\begin{bmatrix}
\begin{array}{l} A_0^T(t)P(t) - 2P(t)B(t)B^T(t)P(t) \\ - 2C^T(t)Q(t)C(t)B(t)B^T(t)P(t) \\ + \dot{P}(t) + P(t)A_0(t) \\ - 2P(t)B(t)B^T(t)C^T(t)Q(t)C(t) \\ + \dot{C}^T(t)Q(t)C(t) + A_0^T(t)C^T(t)Q(t)C(t) \\ - 2C^T(t)Q(t)C(t)B(t)B^T(t)C^T(t)Q(t)C(t) \\ + C^T(t)\dot{Q}(t)C(t) + C^T(t)Q(t)\dot{C}(t) \\ + C^T(t)Q(t)C(t)A_0(t) + 2I \end{array} & \begin{array}{c} P(t)A_1(t) + \\ C^T(t)Q(t)C(t)A_1(t) \end{array} \\
\begin{array}{c} A_1^T(t)P(t) + \\ A_1^T(t)C^T(t)Q(t)C(t) \end{array} & -2I
\end{bmatrix} z(t)
$$

where $z(t) = \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}$. If the condition $2'$ holds, again we end up with $\dot{V}(t) < 0$, which implies that system(1) with controller $u(t) = -\left(B^T(t)P(t) + B^T(t)C^T(t)Q(t)C(t)\right)x(t)$ would be asymptotically stable.

Corollary 2: Consider the following linear time-varying time delay system:

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t-h) + B(t)u(t) \tag{8}$$

$$x(t) = \varphi(t) \text{ ,} -h \le t \le 0$$

The system(8) with the controller $u(t) = -B^T(t)P(t)$ is stable if there are positive definite symmetric matrix $P(t)$ and a positive definite matrix $K(t)$ such that one of the two following conditions holds:

Condition 1:

$$\dot{P}(t) + A_0^T(t)P(t) + P(t)A_0(t) + P(t)A_1(t)A_1^T(t)P(t) - 2P(t)B(t)B^T(t)P(t) + 2I + K(t) = 0$$

Condition 2:

$$
\begin{bmatrix}
\begin{array}{l} \dot{P}(t) + A_0^T(t)P(t) + P(t)A_0(t) \\ - 2P(t)B(t)B^T(t)P(t) + 2I \end{array} & P(t)A_1(t) \\
A_1^T(t)P(t) & -2I
\end{bmatrix} < 0
$$

Corollary 3: Consider the following linear time-invariant system with time delay:

$$\dot{x}(t) = A_0x(t) + A_1x(t-h) + Bu(t) \tag{9-a}$$

$$y(t) = Cx(t) \tag{9-b}$$

$$x(t) = \varphi(t) \text{ ,} -h \le t \le 0$$

The system(9) with the controller $u(t) = -(B^TP + B^TC^TQC)x(t)$ is stable if there are positive definite symmetric matrices $P_{n\times n}, Q_{m\times m}$ and a positive definite matrix $K_{n\times n}$ such that one of the two following conditions holds:

Condition 1:

$$
\begin{aligned}
&A_0^TP + PA_0 + PA_1A_1^TP + A_0^TC^TQC - 2PBB^TP \\
&- 2PBB^TC^TQC - C^TQCBB^TP - 2C^TQCBB^TC^TQC \\
&- C^TQCBB^TP + C^TQCA_0 + C^TQCA_1A_1^TC^TQC + 2I + K = 0
\end{aligned}
$$

Condition 2:

$$\begin{bmatrix} A_0^T P + PA_0 + A_0^T C^T QC - 2PBB^T P \\ - C^T QCBB^T C^T P - 2PBB^T C^T QC - \\ 2C^T QCBB^T C^T QC - C^T QBB^T P \\ + C^T QCA_0 + 2I & PA_1 + C^T QCA_1 \\ A_1^T P + A_1^T C^T QC & -2I \end{bmatrix} < 0$$

3. Extension to multi-delay system

Now we will extend the case to multi-delay system. Consider the following linear time-varying system:

$$\dot{x}(t) = A_0(t)x(t) + \sum_{i=1}^{L} A_i(t)x(t-h_i) + B(t)u(t) \qquad (10-a)$$

$$y(t) = C(t)x(t) \qquad (10-b)$$

$$x(t) = \varphi(t)\ ,\ \bar{h} \le t \le 0$$

$$\bar{h} = \max\{h_i\}, h_i > 0\ , 1 \le i \le L$$

where $x(t) \in \Re^n, u(t) \in \Re^r, y(t) \in \Re^m$.

$A_0(t), A_i(t), B(t), C(t), i = 1,\dots,L$ are continuous matrices with appropriate dimensions. $C(t)$ is differentiable over the interval $[0,\infty)$. h is the constant delay of the system. The initial value $\varphi(t)$ is continuous function in the interval $[-h,0]$. L is the number of the delays.

The following assumption is made for system(10):

Assumption2: The system(10) is stabilizable.

3-1. Stability

Consider the system:

$$\dot{x}(t) = A_0(t)x(t) + \sum_{i=1}^{L} A_i(t)x(t-h_i) \qquad (11-a)$$

$$y(t) = C(t)x(t) \qquad (11-b)$$

$$x(t) = \varphi(t)\ ,\ \bar{h} \le t \le 0$$

$$\bar{h} = \max\{h_i\}, h_i > 0\ , 1 \le i \le L$$

where $x(t) \in \Re^n, u(t) \in \Re^r, y(t) \in \Re^m$.

$A_0(t), A_i(t), B(t), C(t), i = 1,\dots,L$ are continuous matrices with appropriate dimensions. $C(t)$ is differentiable over the interval $[0,\infty)$. h is the constant delay of the system. The initial value $\varphi(t)$ is continuous function in the interval $[-h,0]$. L is the number of the delays.

Theorem 3: The system(11) is stable if there are positive definite symmetric matrices $P_{n\times n}(t), Q_{m\times m}(t)$ and a positive definite matrix $R_{n\times n}(t)$ such that one of the following two conditions holds:

$(Condition1'')$:

$$\dot{P}(t) + A_0^T(t)P(t) + P(t)A_0(t) + \sum_{i=1}^{L} P(t)A_i(t)A_i^T(t)P(t)$$
$$+ \dot{C}^T(t)Q(t)C(t) + C^T(t)\dot{Q}(t)C(t) + A_0^T(t)C^T(t)Q(t)C(t)$$
$$+ C^T(t)Q(t)\dot{C}(t) + C^T(t)Q(t)C(t)A_0(t) +$$
$$\sum_{i=1}^{L} C^T(t)Q(t)C(t)A_i(t)A_i^T(t)C^T(t)Q(t)C(t) + 2LI + R(t) = 0$$

$(Condition2'')$:

$$\begin{bmatrix} \chi & \alpha_1 & \alpha_2 & \cdots & \alpha_L \\ \alpha_1^T & -2I & 0 & \cdots & 0 \\ \alpha_2^T & 0 & -2I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_L^T & 0 & 0 & \cdots & -2I \end{bmatrix} < 0$$

where

$$\chi = A_0^T(t)P(t) + \dot{P}(t) + P(t)A_0(t) +$$
$$\dot{C}^T(t)Q(t)C(t) + A_0^T(t)C^T(t)Q(t)C(t)$$
$$+ C^T(t)\dot{Q}(t)C(t) + C^T(t)Q(t)\dot{C}(t) +$$
$$C^T(t)Q(t)C(t)A_0(t) + 2LI$$

$$\alpha_i = P(t)A_i(t) + C^T(t)Q(t)C(t)A_i(t) \quad , \quad 1 \le i \le L$$

Proof: Consider the following Lyapunov-Krasovskii candidate functional:

$$V(t) = x^T(t)P(t)x(t) + y^T(t)Q(t)y(t) + 2\sum_{i=1}^{L} \int_{t-h_i}^{t} \|x(\tau)\|^2 d\tau$$

Taking it's derivation along the trajectory (11) and using $\dot{y} = \dot{C}x + C\dot{x}$,we get

$$\dot{V} = x^T A_0^T Px + \sum_{i=1}^{L} x^T(t-h_i)A_i^T Px + x^T \dot{P}x +$$
$$x^T PA_0 x + \sum_{i=1}^{L} x^T PA_i x(t-h_i) + x^T \dot{C}^T QCx +$$
$$x^T A_0^T C^T QCx + \sum_{i=1}^{L} x^T(t-h_i)A_i^T C^T QCx$$

$$+x^T C^T \dot{Q} C x + x^T C^T Q \dot{C} x + x^T C^T Q C A_0 x +$$
$$x^T C^T Q C A_1 x(t-h) + 2L\|x\|^2 - 2\sum_{i=1}^{L}\|x(t-h_i)\|^2 \quad (12)$$

Using the lemma, we have

$$\dot{V}_1(t) \le x^T(t)(\dot{P}(t) + A_0^T(t)P(t) + P(t)A_0(t)$$
$$+\sum_{i=1}^{L} P(t)A_i(t)A_i^T(t)P(t) + \dot{C}^T(t)Q(t)C(t) +$$
$$C^T(t)\dot{Q}(t)C(t) + A_0^T(t)C^T(t)Q(t)C(t) +$$
$$C^T(t)Q(t)\dot{C}(t) + C^T(t)Q(t)C(t)A_0(t) +$$
$$\sum_{i=1}^{L} C^T(t)Q(t)C(t)A_1(t)A_1^T(t)C^T(t)Q(t)C(t) + 2LI)x(t)$$

If condition $1''$ holds, we have

$$\dot{V} \le -x^T(t)K(t)x(t) < 0$$

That implies system(11) is asymptotically stable in finite time.

The equality (12) can be rewritten in the following form:

$$\dot{V}(t) = \omega^T(t)\begin{bmatrix} \chi & \alpha_1 & \alpha_2 & \cdots & \alpha_L \\ \alpha_1^T & -2I & 0 & \cdots & 0 \\ \alpha_2^T & 0 & -2I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_L^T & 0 & 0 & \cdots & -2I \end{bmatrix}\omega(t)$$

where

$$\chi = A_0^T(t)P(t) + \dot{P}(t) + P(t)A_0(t) +$$
$$\dot{C}^T(t)Q(t)C(t) + A_0^T(t)C^T(t)Q(t)C(t)$$
$$+C^T(t)\dot{Q}(t)C(t) + C^T(t)Q(t)\dot{C}(t) +$$
$$C^T(t)Q(t)C(t)A_0(t) + 2LI$$
$$\alpha_i = P(t)A_i(t) + C^T(t)Q(t)C(t)A_i(t) \quad , \quad 1 \le i \le L$$
$$\omega(t) = \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}$$

If the condition $2''$ holds, we have $\dot{V}(t) < 0$, which implies that the system(11) is asymptotically stable.

Corollary 4: Consider the following linear time-invariant system with multi-point delays:

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{L} A_i x(t-h_i) \quad (13-a)$$
$$y(t) = Cx(t) \quad (13-b)$$
$$x(t) = \varphi(t), -\bar{h} \le t \le 0$$
$$\bar{h} = \max\{h_i\}, h_i > 0, i = 1, ..., L$$

The system(13) is stable if there are positive definite symmetric matrices $P_{n\times n}, Q_{m\times m}$ and a positive definite matrix $R_{n\times n}$ such that one of the following two conditions holds:

Condition 1:

$$A_0^T P + PA_0 + \sum_{i=1}^{L} PA_i A_i^T P + A_0^T C^T QC +$$
$$C^T QCA_0 + C^T QCA_1 A_1^T C^T QC + 2LI + R = 0$$

Condition 2:

$$\begin{bmatrix} \chi & \alpha_1 & \alpha_2 & \cdots & \alpha_L \\ \alpha_1^T & -2I & 0 & \cdots & 0 \\ \alpha_2^T & 0 & -2I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_L^T & 0 & 0 & \cdots & -2I \end{bmatrix} < 0$$

where

$$\chi = A_0^T P + PA_0 + A_0^T C^T QC + C^T QCA_0 + 2LI$$
$$\alpha_i = PA_i + C^T QCA_i$$

3-2. Stabilization

Theorem4: The system(10) with condition (**A2**) and the controller $u(t) = -(B^T(t)P(t) + B^T(t)C^T(t)Q(t)C(t))x(t)$ is stable if there are positive definite symmetric matrices $P_{n\times n}(t), Q_{m\times m}(t)$ and a positive definite matrix $R_{n\times n}(t)$ such that one of the following two conditions holds:

(*Condition* $1'''$):

$$\dot{P}(t) + A_0^T(t)P(t) + P(t)A_0(t) + \sum_{i=1}^{L} P(t)A_i(t)A_i^T(t)P(t)$$
$$-2P(t)B(t)B^T(t)P(t) - 2P(t)B(t)B^T(t)C^T(t)Q(t)C(t)$$
$$-2C^T(t)Q(t)C(t)B(t)B^T(t)P(t) + \dot{C}^T(t)Q(t)C(t) +$$
$$C^T(t)\dot{Q}(t)C(t) + A_0^T(t)C^T(t)Q(t)C(t) + C^T(t)Q(t)\dot{C}(t)$$
$$-2C^T(t)Q(t)C(t)B(t)B^T(t)C^T(t)Q(t)C(t)$$
$$+C^T(t)Q(t)C(t)A_0(t) +$$
$$\sum_{i=1}^{L} C^T(t)Q(t)C(t)A_1(t)A_1^T(t)C^T(t)Q(t)C(t)$$
$$+2LI + R(t) = 0$$

$(Condition\ 2''')$:

$$\begin{bmatrix} \chi & \alpha_1 & \alpha_2 & \cdots & \alpha_L \\ \alpha_1^T & -2I & 0 & \cdots & 0 \\ \alpha_2^T & 0 & -2I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_L^T & 0 & 0 & \cdots & -2I \end{bmatrix} < 0$$

where

$$\begin{aligned} \chi &= A_0^T(t)P(t) - 2P(t)B(t)B^T(t)P(t) \\ &- 2C^T(t)Q(t)C(t)B(t)B^T(t)P(t) \\ &+ \dot{P}(t) + P(t)A_0(t) \\ &- 2P(t)B(t)B^T(t)C^T(t)Q(t)C(t) \\ &+ \dot{C}^T(t)Q(t)C(t) + A_0^T(t)C^T(t)Q(t)C(t) \\ &- 2C^T(t)Q(t)C(t)B(t)B^T(t)C^T(t)Q(t)C(t) \\ &+ C^T(t)\dot{Q}(t)C(t) + C^T(t)Q(t)\dot{C}(t) \\ &+ C^T(t)Q(t)C(t)A_0(t) + 2LI \end{aligned}$$

$$\alpha_i = P(t)A_i(t) + C^T(t)Q(t)C(t)A_i(t) \qquad , \ 1 \le i \le L$$

Note that this theorem is a generalized form of the results in [2].

Proof: Consider the following Lyapunov-Krasovskii functional candidate:

$$V(t) = x^T(t)P(t)x(t) + y^T(t)Q(t)y(t) + 2\sum_{i=1}^{L}\int_{t-h_i}^{t}\|x(\tau)\|^2 d\tau$$

Taking it's derivation along trajectory (10) and using $\dot{y} = \dot{C}x + C\dot{x}$,we have

$$\begin{aligned} \dot{V} &= \left(A_0x + \sum_{i=1}^{L}A_ix(t-h_i) + Bu\right)^T Px + x^T\dot{P}x + \\ & x^T P\left(A_0x + \sum_{i=1}^{L}A_ix(t-h_i) + Bu\right) + \\ & \left(\dot{C}x + C\left(A_0x + \sum_{i=1}^{L}A_ix(t-h_i) + Bu\right)\right)^T QCx + x^TC^T\dot{Q}Cx + \\ & x^TC^TQ\left(\dot{C}x + C\left(A_0x + \sum_{i=1}^{L}A_ix(t-h_i) + Bu\right)\right) \\ & + 2L\|x\|^2 - 2\sum_{i=1}^{L}\|x(t-h_i)\|^2 \end{aligned}$$

Using $u = -\left(B^TP + B^TC^TQC\right)x$,we get

$$\begin{aligned} \dot{V} &= \left(A_0x + A_1x(t-h) - B\left(B^TP + B^TC^TQC\right)x\right)^T Px + \\ & x^T\dot{P}x + x^TP\left(A_0x + A_1x(t-h) - B\left(B^TP + B^TC^TQC\right)x\right) + \\ & \left(\dot{C}x + C\left(A_0x + A_1x(t-h) - B\left(B^TP + B^TC^TQC\right)x\right)\right)^T QCx \\ & + x^TC^T\dot{Q}Cx + x^TC^TQ\left(\dot{C}x + C\left(A_0x + A_1x(t-h) - B\left(B^TP + B^TC^TQC\right)x\right)\right) \\ & + 2\|x\|^2 - 2\|x(t-h)\|^2 = \\ & x^TA_0^TPx + x^T(t-h)A_1^TPx - x^TPBB^TPx - \\ & x^TC^TQCBB^TPx + x^T\dot{P}x + x^TPA_0x + x^T\dot{C}^TQCx + \\ & x^TA_0^TC^TQCx + x^T(t-h)A_1^TC^TQCx - x^TPBB^TC^TQCx - \\ & x^TC^TQCBB^TC^TQCx + x^TC^T\dot{Q}Cx + x^TC^TQ\dot{C}x + x^TC^TQCA_0x + \\ & x^TPA_1x(t-h) - x^TPBB^TPx - x^TPBB^TC^TQCx + \\ & x^TC^TQCA_1x(t-h) - x^TC^TQBB^TPx - x^TC^TQBB^TC^TQCx + \\ & 2\|x\|^2 - 2\|x(t-h)\|^2 \end{aligned} \tag{14}$$

Using the lemma, the equality (14) is

$$\begin{aligned} \dot{V}(t) &\le x^T(t)(\dot{P}(t) + A_0^T(t)P(t) + P(t)A_0(t) + \\ & \sum_{i=1}^{L}P(t)A_i(t)A_i^T(t)P(t) - 2P(t)B(t)B^T(t)P(t) - \\ & 2P(t)B(t)B^T(t)C^T(t)Q(t)C(t) + C^T(t)Q(t)\dot{C}(t) \\ & - 2C^T(t)Q(t)C(t)B(t)B^T(t)P(t) + \dot{C}^T(t)Q(t)C(t) + \\ & C^T(t)\dot{Q}(t)C(t) + A_0^T(t)C^T(t)Q(t)C(t) \\ & - 2C^T(t)Q(t)C(t)B(t)B^T(t)C^T(t)Q(t)C(t) \\ & + C^T(t)Q(t)C(t)A_0(t) + \\ & \sum_{i=1}^{L}C^T(t)Q(t)C(t)A_1(t)A_1^T(t)C^T(t)Q(t)C(t) \\ & + 2LI)x(t) = 0 \end{aligned} \tag{15}$$

If the condition $1'''$ holds, equality(15) is $\dot{V} \le -x^T(t)R(t)x(t) < 0$,which implies that the system(10) with controller $u(t) = -\left(B^T(t)P(t) + B^T(t)C^T(t)Q(t)C(t)\right)x(t)$ would be asymptotically stable in finite time.

The equality (14) can be rewritten as follows:

$$\vartheta^T(t)\begin{bmatrix} \chi & \alpha_1 & \alpha_2 & \cdots & \alpha_L \\ \alpha_1^T & -2I & 0 & \cdots & 0 \\ \alpha_2^T & 0 & -2I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_L^T & 0 & 0 & \cdots & -2I \end{bmatrix}\vartheta(t)$$

where

$$\chi = A_0^T(t)P(t) - 2P(t)B(t)B^T(t)P(t)$$
$$- 2C^T(t)Q(t)C(t)B(t)B^T(t)P(t)$$
$$+ \dot{P}(t) + P(t)A_0(t)$$
$$- 2P(t)B(t)B^T(t)C^T(t)Q(t)C(t)$$
$$+ \dot{C}^T(t)Q(t)C(t) + A_0^T(t)C^T(t)Q(t)C(t)$$
$$- 2C^T(t)Q(t)C(t)B(t)B^T(t)C^T(t)Q(t)C(t)$$
$$+ C^T(t)\dot{Q}(t)C(t) + C^T(t)Q(t)\dot{C}(t)$$
$$+ C^T(t)Q(t)C(t)A_0(t) + 2LI$$
$$\alpha_i = P(t)A_i(t) + C^T(t)Q(t)C(t)A_i(t) \quad , 1 \le i \le L$$
$$\theta(t) = [x(t), x(t-h_1), \ldots, x(t-h_L)]^T$$

If the condition $2'''$ holds, we have $\dot{V}(t) < 0$,which implies that the system(10) with controller $u(t) = -\left(B^T(t)P(t) + B^T(t)C^T(t)Q(t)C(t)\right)x(t)$ is asymptotically stable.

Corollary 5: Consider the following linear time-varying system with multi point delays:

$$\dot{x}(t) = A_0(t)x(t) + \sum_{i=1}^{L} A_i(t)x(t-h) + B(t)u(t) \qquad (16)$$
$$x(t) = \varphi(t) \ , -\bar{h} \le t \le 0$$
$$\bar{h} = \max\{h_i\}, h_i > 0, \ i = 1, \ldots, L$$

The system(16) with controller $u(t) = -B^T(t)P(t)$ is stable if there are positive definite symmetric matrices $P_{n\times n}(t), Q_{m\times m}(t)$ and a positive definite matrix $R_{n\times n}(t)$ such that one of the following two conditions holds:

Condition 1:

$$\dot{P}(t) + A_0^T(t)P(t) + P(t)A_0(t) + \sum_{i=1}^{L} P(t)A_i(t)A_i^T(t)P(t)$$
$$- 2P(t)B(t)B^T(t)P(t) + 2LI + R(t) = 0$$

Condition 2:

$$\begin{bmatrix} \chi & \alpha_1 & \alpha_2 & \cdots & \alpha_L \\ \alpha_1^T & -2I & 0 & \cdots & 0 \\ \alpha_2^T & 0 & -2I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_L^T & 0 & 0 & \cdots & -2I \end{bmatrix} < 0$$

where

$$\chi = \dot{P}(t) + A_0^T(t)P(t) + P(t)A_0(t) -$$
$$2P(t)B(t)B^T(t)P(t) + 2LI$$
$$\alpha_i = P(t)A_i(t) \ , \ 1 \le i \le L$$

It is clear that these results are the same as [2]. It shows that the results of [2] are a special case of our results.

Corollary 6: Consider the following linear time-invariant system with multiple delays:

$$\dot{x}(t) = A_0x(t) + \sum_{i=1}^{L} A_ix(t-h) + Bu(t) \qquad (17-a)$$
$$y(t) = Cx(t) \qquad (17-b)$$
$$x(t) = \varphi(t) \ , -\bar{h} \le t \le 0$$
$$\bar{h} = \max\{h_i\}, h_i > 0, \ i = 1, \ldots, L$$

The system(17) with controller $u(t) = -(B^TP + B^TC^TQC)x(t)$ is stable if there are positive definite symmetric matrices $P_{n\times n}, Q_{m\times m}$ and a positive definite matrix $R_{n\times n}$ such that one of the following two conditions holds:

Condition 1:

$$A_0^TP + PA_0 + PA_1A_1^TP + A_0^TC^TQC - 2PBB^TP$$
$$- 2PBB^TC^TQC - C^TQCBB^TP$$
$$- 2C^TQCBB^TC^TQC - C^TQCBB^TP + C^TQCA_0$$
$$+ \sum_{i=1}^{L} C^TQCA_iA_i^TC^TQC + 2LI + R = 0$$

Condition 2:

$$\begin{bmatrix} \chi & \alpha_1 & \alpha_2 & \cdots & \alpha_L \\ \alpha_1^T & -2I & 0 & \cdots & 0 \\ \alpha_2^T & 0 & -2I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_L^T & 0 & 0 & \cdots & -2I \end{bmatrix} < 0$$

where

$$\chi = A_0^TP + PA_0 + A_0^TC^TQC - 2PBB^TP$$
$$- C^TQCBB^TC^TP - 2PBB^TC^TQC -$$
$$2C^TQCBB^TC^TQC - C^TQBB^TP + C^TQCA_0 + 2LI$$
$$\alpha_i = PA_i + C^TQCA_i$$

4. Numerical example

Consider the system:

$$\dot{x}(t)=\begin{pmatrix} e^{6t} & 0 \\ 0 & 1 \end{pmatrix}x(t)+\begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{pmatrix}x(t-2)+\begin{pmatrix} 2e^{t} & 0 \\ 0 & -1 \end{pmatrix}u(t)$$

$$y(t)=\begin{pmatrix} e^{t} & 0 \\ 0 & 1 \end{pmatrix}x(t)$$

$$x(t)=[2,2]^{T} \quad for \quad -2\le t\le 0$$

It can be verified easily that both states and outputs of the system are unstable.

We choose the following matrices that satisfy the condition1 of theorem 2 which implies the asymptotically stabilization of the system:

$$P(t)=\begin{bmatrix} \frac{1}{4}e^{4t} & 0 \\ 0 & 1 \end{bmatrix}, Q(t)=\begin{bmatrix} e^{-4t} & 0 \\ 0 & 2+e^{-t} \end{bmatrix}$$

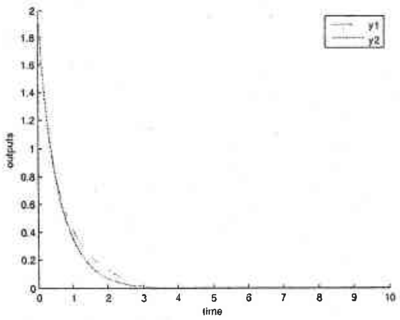

Fig.1 the outputs of the system

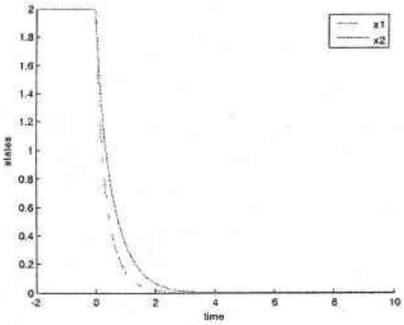

Fig.2 the states of the system

We have simulated this system with the controller we designed and the results are illustrated by Fig.1 and 2. These figures show that the both states and outputs of the system are stable.

5. Conclusions

In this paper, we considered both stability and stabilization problems of a class of linear time-varying systems with time delay. We designed a new controller for the stabilization of this class of systems. It can be verified that the considered system may be unstable if the system's states whole are stable. A numerical example was provided in order to show the results established.

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