Electronic III

The Stability Problem

The Stability Problem

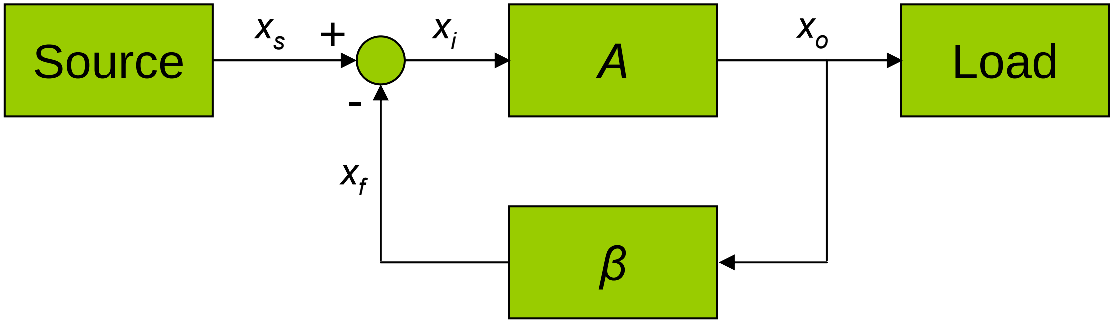


Fig. 4.1

- *A*(s) is the open-loop transfer function. *β* the feedback factor is normally resistive, but this need not be always the case, so generally *β*(s) is the feedback transfer function.

$$A_f(s) = \frac{A(s)}{1 + A(s)\beta(s)} \qquad (4-1)$$

For physical frequency $s = j\omega$ then

$$A_f(j\omega) = \frac{A(j\omega)}{1 + A(j\omega)\beta(j\omega)} \qquad (4-2)$$

The Stability Problem

- Loop-gain is a complex number represented by its magnitude and phase:

$$L(j\omega) = A(j\omega)\,\beta(j\omega) = \left| A(j\omega)\,\beta(j\omega) \right| e^{j\varphi(\omega)} \qquad (4-3)$$

- The manner in which the loop-gain varies with freq. determines the stability or instability of the feedback amplifier.
- Consider the freq. at which $\varphi(\omega)$ becomes 180°. At this frequency, ω_{180}, the loop gain $A(j\omega)\ \beta(j\omega)$ is negative real number, so at this freq. feedback becomes positive.
- If at ω_{180} the magnitude of the loop gain is unity, $A_f(j\omega)$ will be infinite, i.e. the amplifier has an output for zero input; this is by definition an *oscillator*.
- Consider Fig. 4.1, with x_s set to zero, any disturbance in the circuit, such as turning on the power supply, will generate a signal $x_i(s)$ at the input of the amplifier. This signal usually contains a wide range of frequencies, including a component at $\omega = \omega_{180}$, that is $x_i \sin(\omega_{180} t)$, so:

$$X_f = \underbrace{A(j\omega_{180})\,\beta(j\omega_{180})}_{-1}\,x_i = -x_i \qquad (4-4)$$

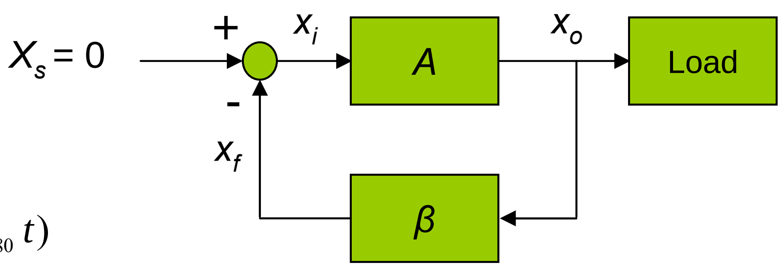
The Stability Problem

- x_f is further multiplied by -1 in summer, so the feedback causes the signal x_i to be sustained. We started from $x_i \sin(\omega_{180}\, t)$ and resulted to the same $x_i \sin(\omega_{180}\, t)$ at the same point so the amplifier is said to oscillate at the freq. ω_{180} .
- For the case where the magnitude of the loop is greater than unity:

  the circuit will oscillate and oscillation grows in amplitude until some nonlinearity (always present in circuit) reduces the magnitude of the loop gain to exactly unity, at which sustained oscillation will be obtained.

  for $A(j\omega)\ \beta(j\omega) = -2$ for example if we start from $x_i \sin(\omega_{180}\, t)$ around the loop we get -2 x_i, so after summer we obtain 2 $x_{i.}$ After one more iteration 4 x_i will be produced. Traveling more around the loop will grow the signal further 8 x_i , 16 x_i , …
- <u>Our objective here is to prevent this oscillation!!!</u>

The Stability Problem

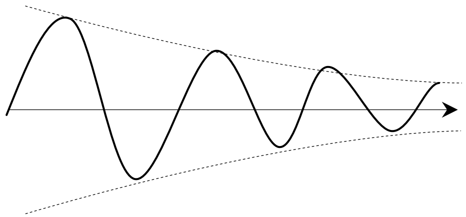


$$A(j\omega_{180})\,\beta(j\omega_{180}) = -\frac{1}{2}$$

$$x_i = A\sin(\omega_{180}\,t) \quad\Rightarrow\quad x_f = -0.5\,A\sin(\omega_{180}\,t)$$

$$x_i = 0.5A\sin(\omega_{180}\,t) \quad\Rightarrow\quad x_f = -0.25\,A\sin(\omega_{180}\,t)$$

$$x_i = 0.25A\sin(\omega_{180}\,t) \quad\Rightarrow\quad x_f = -0.125\,A\sin(\omega_{180}\,t)$$



$$A(j\omega_{180})\,\beta(j\omega_{180}) = -1$$

$$x_i = A\sin(\omega_{180}\,t) \quad\Rightarrow\quad x_f = -\,A\sin(\omega_{180}\,t)$$

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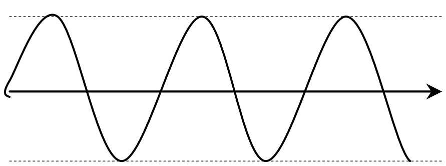
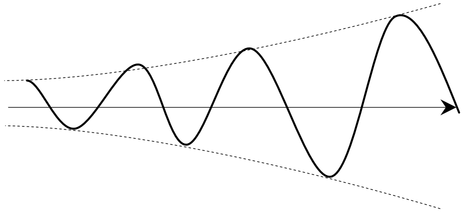


$$A(j\omega_{180})\,\beta(j\omega_{180}) = -2$$

$$x_i = A\sin(\omega_{180}\,t) \quad\Rightarrow\quad x_f = -2\,A\sin(\omega_{180}\,t)$$

$$x_i = 2A\sin(\omega_{180}\,t) \quad\Rightarrow\quad x_f = -4\,A\sin(\omega_{180}\,t)$$

$$x_i = 4A\sin(\omega_{180}\,t) \quad\Rightarrow\quad x_f = -8\,A\sin(\omega_{180}\,t)$$



The Nyquist Plot

- It is a formalized approach for testing the stability based on the above discussion. It is a polar plot of the loop gain with freq. used as a parameter.
- For transfer function like

$$\frac{A_0\beta}{(1+\frac{s}{p_1})(1+\frac{s}{p_2})\ldots} \qquad (4-5)$$

in which β is assumed positive (i.e. negative feedback with negative sign as shown in fig. 4.1).

$$\varphi(\omega) = -\left(\tan^{-1}\frac{\omega}{p_1} + \tan^{-1}\frac{\omega}{p_1} + \ldots\right) \qquad (4-6)$$

For positive frequencies ($+\omega$), the phase of the loop gain will be negative as shown by solid-line in fig. 4.2.

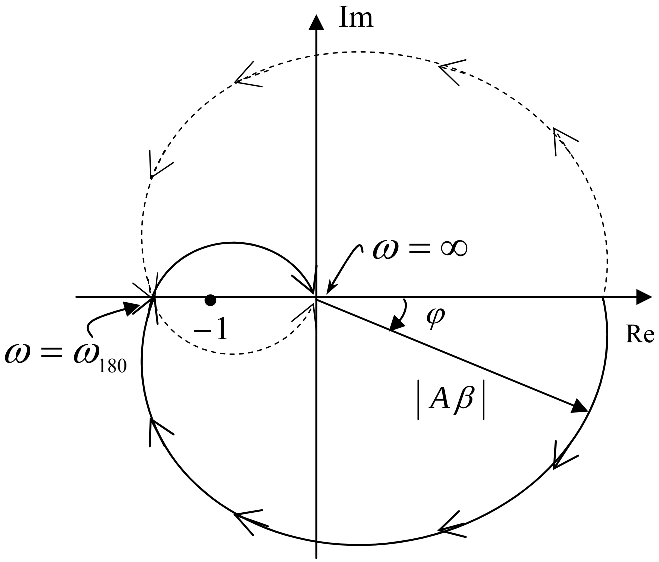


Fig. 4.2. The Nyquist plot

The Nyquist Plot

- Note that the radial distance is $|A\beta|$ and the angle is φ. The solid-line is for positive frequencies. Since the gain function of any physical network has a magnitude that is even function of freq. and a phase that is odd, the $A\beta$ plot for negative freq. is mirror image through the Re-axis.
- ω_{180} is the intersect point of Nyquist plot and Re-axis. If this happens to the left of (-1,0), so the magnitude of loop gain is greater than unity and amplifier will be unstable. But if the intersection happens to the right of (-1,0), the amplifier will be stable.
- If the Nyquist plot encircles (-1,0) the amplifier will be unstable. The same as saying that the magnitude of loop gain is greater than unity @ intersection point.

The Nyquist Plot

Example:

$$A(s)=\left(\frac{10}{1+s/10^4}\right)^3 \quad \rightarrow \quad \omega_{180}=?\,,\ \beta_{cr}\,(\text{where } A_f \text{ is stable})=?$$

Solution:

$$A(j\omega)=\left(\frac{10}{1+j\omega/10^4}\right)^3=\left(\frac{10}{1+j\omega/10^4}\right)\left(\frac{10}{1+j\omega/10^4}\right)\left(\frac{10}{1+j\omega/10^4}\right)$$

$$\Rightarrow \quad \varphi(\omega)=-3\tan^{-1}\left(\omega/10^4\right)$$

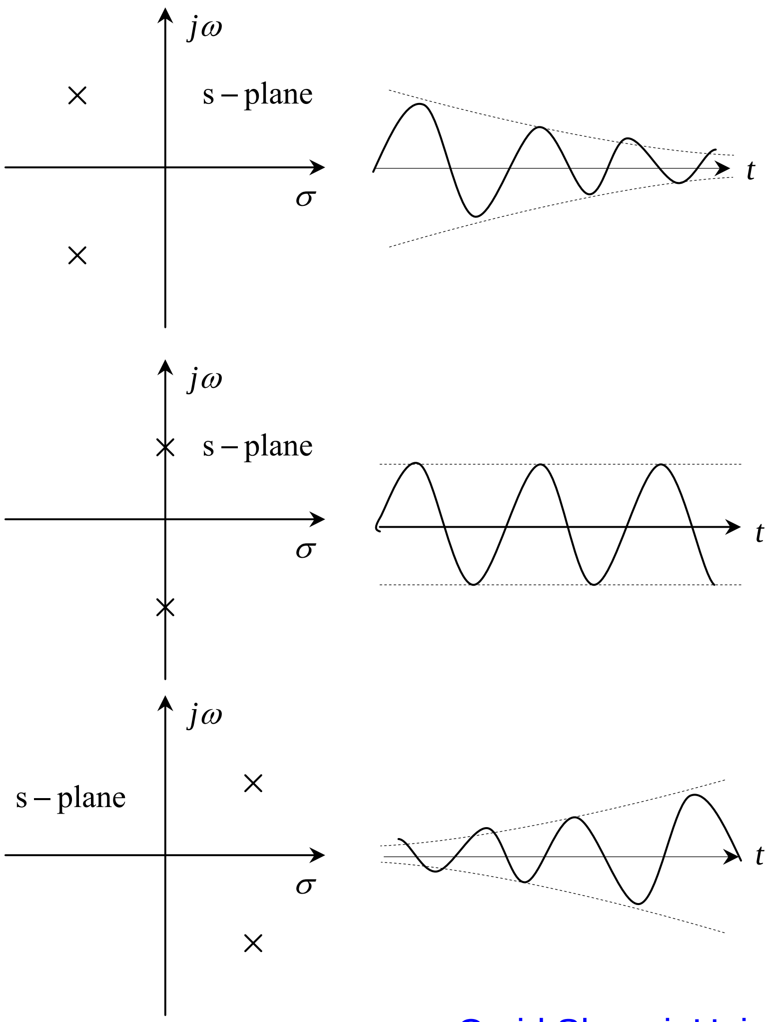
$$@ \quad \omega_{180}\,,\ \varphi=180^\circ \quad \Rightarrow \quad \tan^{-1}\left(\omega/10^4\right)=60^\circ \quad \Rightarrow \quad \omega/10^4=\sqrt{3}$$

$$\Rightarrow \quad \omega_{180}=\sqrt{3}\times 10^4 \ \text{rad/sec}$$

The feedback amp. will be stable if @ ω180 , $|A\beta|<1$. @ the boundary $\beta=\beta_{cr}$:

$$\left|A\beta_{cr}\right|=1\,,\ \left|A(j\omega)\right|=\left(\frac{10}{\sqrt{1+\left(\omega/10^4\right)^2}}\right)^3 \quad \Rightarrow \quad \beta_{cr}=\frac{1}{\left|A(j\omega_{180})\right|}=\frac{1}{\frac{1000}{(1+3)^{3/2}}}=0.008$$

Stability and Pole Location



- An amplifier with a pole pair at $s = \sigma \pm j\omega_n$ can have a response to a disturbance as follows:

$$v(t) = e^{\sigma t}\left[e^{j\omega_n t} + e^{-j\omega_n t}\right] = 2e^{\sigma t}\cos(\omega_n t)$$

- Depending upon the position of complex-pole pair shown in Fig.4.3 we could have damped oscillation (stable) sustained or growing oscillation (unstable) system.

Fig. 4.3

Stability and Pole Location

- **Poles of feedback amp.**
- They can be obtained from equation (4-1):

$$1 + A(s)\,\beta(s) = 0 \qquad (4-7)$$

Which is called the characteristic equation.

- We will see how the feedback affects the amplifier poles. We will assume that the open-loop amplifier has real poles and no finite zeros (all zeros are at $s=\infty$). For simplification here we assume β is independent of frequency too.

Stability and Pole Location

- **Single pole amp:**

$$A(s)=\frac{A_0}{1+s/\omega_p} \qquad (4-8)$$

$$A_f(s)=\frac{A_0/(1+A_0\beta)}{1+s/[\omega_p(1+A_0\beta)]} \qquad (4-9)$$

- So the feedback moves the pole along negative real axis to a freq. ω_{pf}:

$$\omega_{pf}=\omega_p(1+A_0\beta) \qquad (4-10)$$

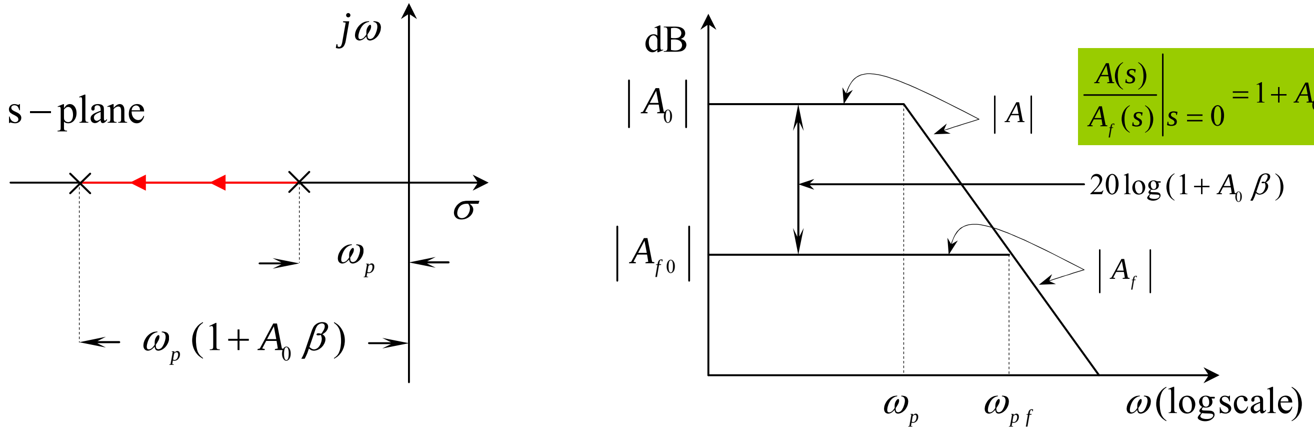


Fig. 4.4. Effect of feedback on (a) pole location, (b) the freq. response having a single-pole

Stability and Pole Location

- Note in the Bode plots although at low freq. while the difference between the two plots is 20 log(1+$A_0\beta$), the two curves coincide at high freq. from (4-9) for frequencies $\omega >> \omega_p(1+A_0\beta)$:

$$A_f(s) \approx \frac{A_0/(1+A_0\beta)}{s/[\omega_p(1+A_0\beta)]} = \frac{A_0\omega_p}{s} \approx A(s) \qquad (4-11)$$

- Physically speaking, the loop gain is much smaller than unity at such high freq. and the feedback is ineffective.
- This amp is <u>unconditionally stable</u>. Since the closed-loop pole never enters RHP of s-plane for any β. The phase lag of a single-pole amplifier can be never greater than 90° so never achieving 180° phase shift required for the feedback to become positive!

Stability and Pole Location

- The Nyquist plot for this system is:

$$A\beta(s) = \frac{A_0\beta}{1 + s/\omega_p} \quad \Rightarrow \quad A\beta(j\omega) = \frac{A_0\beta}{1 + j\omega/\omega_p}$$

$$A\beta(j\omega) = \frac{A_0\beta\left(1 - j\omega/\omega_p\right)}{1 + \left(\omega/\omega_p\right)^2}$$

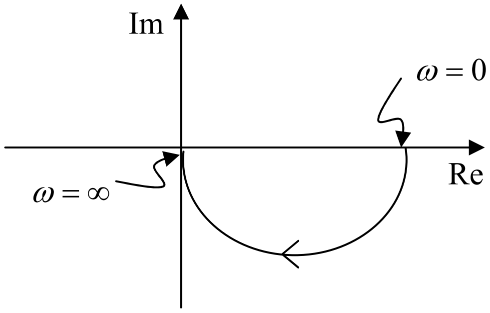


Fig. 4.5

$$\text{Note}: \angle A\beta(j\omega)\Big|_{\omega=0} = 0 \quad \Rightarrow \quad \text{Im}(A\beta(j\omega)) = 0$$

$$\text{For } \omega \to \infty : \begin{cases} \left| A\beta(j\omega) \right| \to 0 \\ \angle A\beta(j\omega) = -\tan^{-1} \dfrac{\omega}{\omega_p}\Bigg|_{\omega \to \infty} = -\dfrac{\pi}{2} \end{cases}$$

Stability and Pole Location

- **Amp. with two-pole response:**
- Consider a system with two real poles:

$$A(s) = \frac{A_0}{\left(1+s/\omega_{p1}\right)\left(1+s/\omega_{p2}\right)} \qquad (4-12)$$

- closed-loop poles are obtained from $1+A(s)\,\beta=0$:

$$s^2 + s\underbrace{\left(\omega_{p1}+\omega_{p1}\right)}_{\frac{\omega_0}{Q}} + \underbrace{(1+A_0\,\beta)\,\omega_{p1}\,\omega_{p2}}_{\omega_0^2} = 0 \qquad (4-13)$$

$$s = -\frac{1}{2}\left(\omega_{p1}+\omega_{p2}\right) \pm \frac{1}{2}\sqrt{(\omega_{p1}+\omega_{p2})^2 - 4\,(1+A_0\,\beta)\,\omega_{p1}\,\omega_{p2}} \qquad (4-14)$$

- So as loop gain $A_0\,\beta$ is increased from zero, the poles are brought closer together. Then a value of loop gain is further increased, the poles become complex conjugate and move along a vertical line. This plot i.e. *root-Locus diagram* is shown in Fig.4.6 .

Stability and Pole Location

- From the root-locus diagram of Fig.4.6 we see that this feedback amp also is unconditionally stable.
- The max. phase shift of *A(s)* in this case is 180° (90° per pole). But this value is retuned at $\omega=\infty$. Thus there is no freq. at which the phase shift reaches 180°!
- Although the open-loop amp *A*(s) may have a dominant pole. This is not necessarily the case for the closed-loop amp.
- Once the poles of closed-loop are known, its response can be plotted.

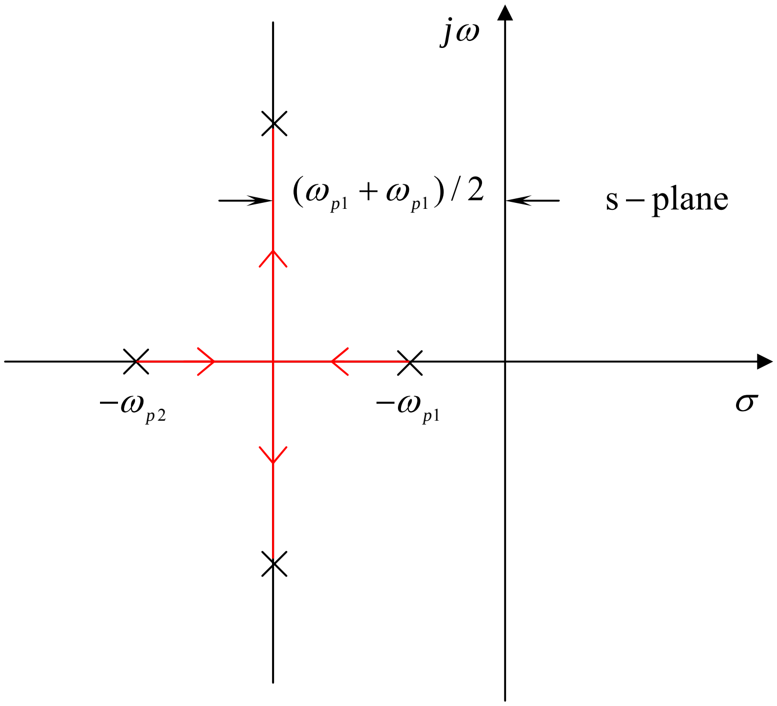


Fig. 4.6

Stability and Pole Location

- The characteristic equation of a second-order network can be written:

$$s^2 + s\frac{\omega_0}{Q} + \omega_0^2 = 0 \qquad (4-15)$$

- Where ω_0 is pole frequency and Q is pole Q factor.
- The poles are complex if Q>0.5. (starting of bringing overshoot in time step response)

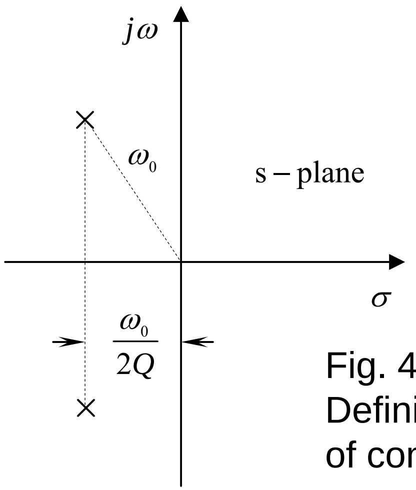


Fig. 4.7
Definition of ω_0 and Q of a pair of complex conjugate poles

- A geometric interpretation for ω_0 and Q of a pair of complex-conjugate poles is given in Fig.4.7. ω_0 is the radial distance of the poles and Q indicates the distance of the poles from $j\omega$-axis (poles on $j\omega$-axis have Q=∞). Comparing (4-13) & (4-15):

$$Q = \frac{\sqrt{(1+A_0\beta)\omega_{p1}\omega_{p2}}}{(\omega_{p1}+\omega_{p2})} \qquad (4-16)$$

$$\omega_0 = \sqrt{(1+A_0\beta)\omega_{p1}\omega_{p2}} \qquad (4-17)$$

Stability and Pole Location

- For $Q \leq 0.707 = \sqrt{2}/2$ the freq. response of the feedback amp. shows no peaking. For Q = 0.707 (poles at 45° angle) results in maximally flat freq. response.

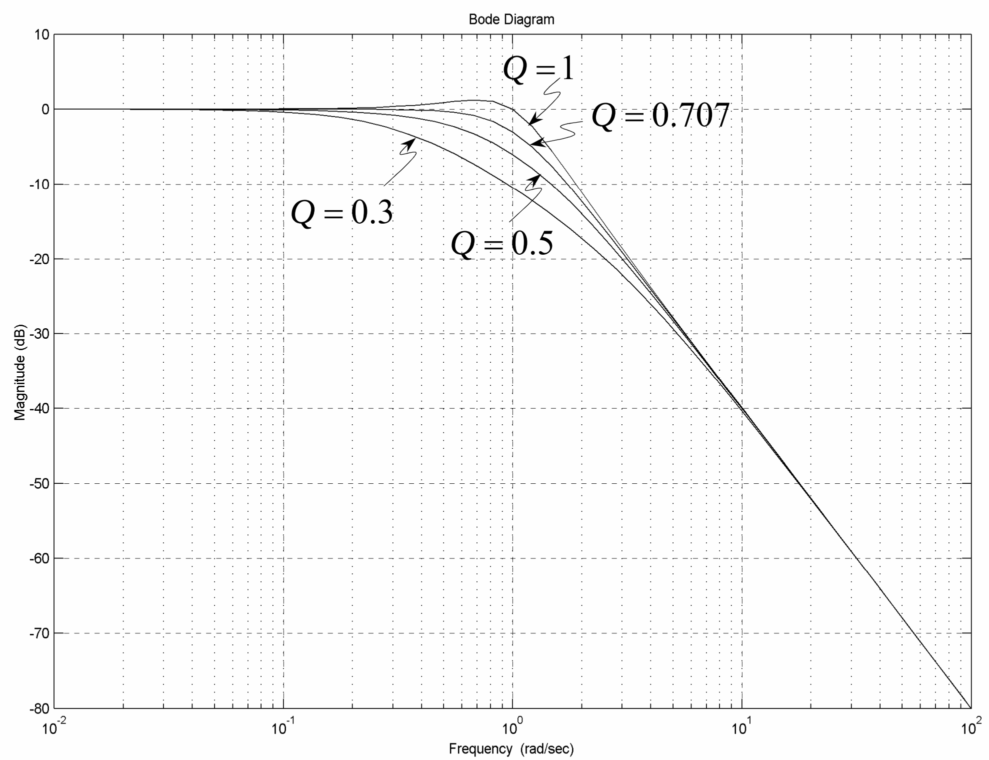


Fig. 4.8.
Shows the feedback amp response for difference Q or various values of $A_0\beta$.

Stability and Pole Location

Example:

Consider positive feedback circuit in Fig.4.9. Sketch its root-locus diagram vs. *k*.

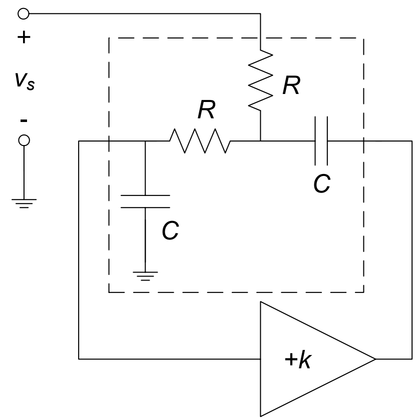
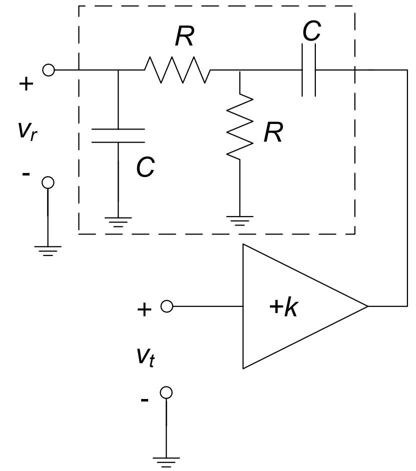


Fig. 4.9

Solution:

To obtain the loop transfer function, signal-source is short circuited and loop is broken at amplifier input, we then apply a test voltage v_t and find the returned voltage v_r (which was connected to the amp input before breaking the loop)!



Stability and Pole Location

$$L(s) = \frac{v_r}{v_t} = kT(s) \qquad (4-18)$$

$T(s)$ is the two-part RC network inside the broken-line box:

$$T(s) = \frac{s/RC}{s^2 + s\ (3/RC) + (1/RC)^2} \qquad (4-19)$$

$$L(s) = \frac{s\ (K/RC)}{s^2 + s\ (3/RC) + (1/RC)^2} \qquad (4-20)$$

Characteristic equation: 1- $L(s)$=0 (4-21)

$$s^2 + s\left(\frac{3}{RC}\right) + \left(\frac{1}{RC}\right)^2 - s\left(\frac{k}{RC}\right) = 0$$

$$\Rightarrow\ s^2 + s\left(\frac{3-K}{RC}\right) + \left(\frac{1}{RC}\right)^2 = 0 \qquad (4-22)$$

Comparing to standard form (4-15):

$$\omega_0 = \frac{1}{RC} \qquad (4-23)$$

$$Q = \frac{1}{3-K} \qquad (4-24)$$

Stability and Pole Location

- For k=0 Q=1/3 $\Rightarrow$ poles on negative real axis!
- As k increased poles get closer and coincide at Q=0.5 & k=1. further increasing k makes complex poles but since ω_0 remains constant independent of k, the root locus is then a circle:

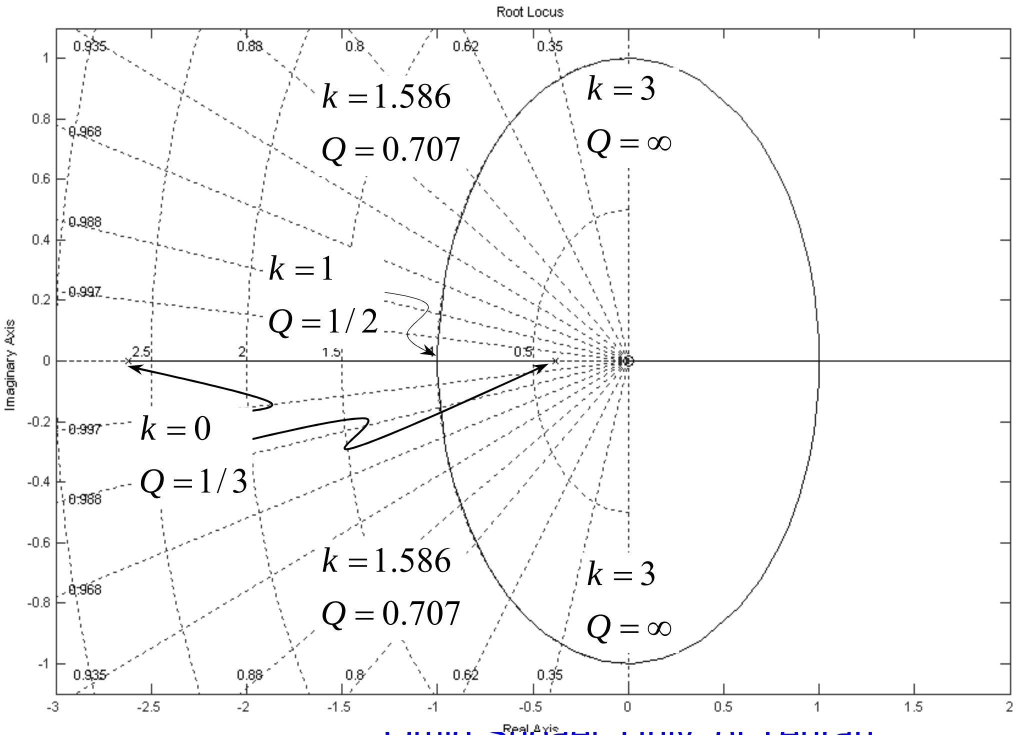


Fig. 4.10

Stability and Pole Location

- For Q=0.707 maximally flat response is obtained $\Rightarrow$ (for k=1.856) which poles are at 45° angle.
- $Q=\infty$ happens when poles cross jω-axis (for k=3).
- Thus for $k \geq 3$ this circuit becomes unstable.

Note: we said a feedback amp with second-order response is unconditionally stable. This circuit, however, is different from negative-feedback. Here we have a positive gain k and a feedback with transfer function $T(s)$. The circuit oscillate at frequencies for which phase of $T(j\omega)$ is zero.

Stability and Pole Location

A Better approach to find Loop Gain:

Insert a voltage source V_x inside the loop ad find V_z/V_d; The Loop Gain $L(s)$, Closed Loop Gain $A_f(s)$ … are obtained while the loading effects are automatically there!

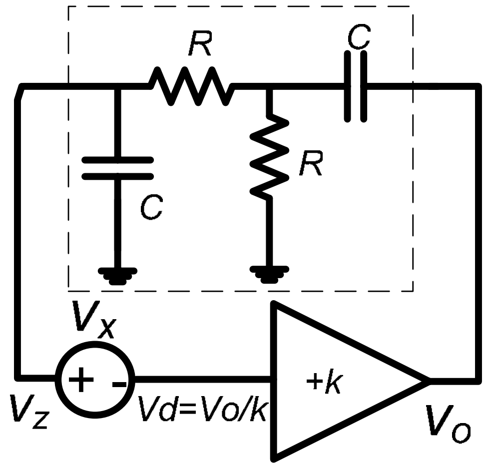


Fig. 4.11

$$\left.\begin{aligned} \frac{V_z}{V_d} &= \frac{V_x + V_d}{V_d} = 1 + \frac{V_x}{V_d} \\ \frac{V_z}{V_d} &= \frac{V_z}{V_o}\frac{V_o}{V_d} = \beta .K \end{aligned}\right\} \rightarrow \frac{V_x}{V_d} = -1 + \beta .K$$

$$\text{also} \quad \frac{V_z}{V_x} = 1 + \frac{V_d}{V_x} = \frac{\beta .K}{-1 + \beta .K} \quad \text{i.e. the Closed Loop Gain!}$$

$$V_z = \frac{V_o}{K} + V_x$$

$$\beta = \frac{s/RC}{s^2 + s\,(3/RC) + (1/RC)^2}$$

$$\Rightarrow \frac{V_z}{V_x} = \frac{\dfrac{s(K/RC)}{s^2 + s\,(3/RC) + (1/RC)^2}}{-1 + \dfrac{Ks/RC}{s^2 + s\,(3/RC) + (1/RC)^2}} = \frac{-s(K/RC)}{s^2 + s\left(\dfrac{3-k}{RC}\right) + \left(\dfrac{1}{RC}\right)^2}$$

Stability and Pole Location

- **Amplifier with three or more poles:**
- Fig.4.12 shows root-locus for a feedback amp whose open-loop has three poles.

  $1+A(s)\beta=0$ (4-25)
- Increasing the loop gain $A(s)\beta$ from zero moves the highest-freq. pole outward and the two other poles are brought closer.
- As $A(s)\beta$ is increased further, two poles become complex conjugate. For a value of $A(s)\beta$ the complex conjugate poles enter RHP causing amp to become unstable.

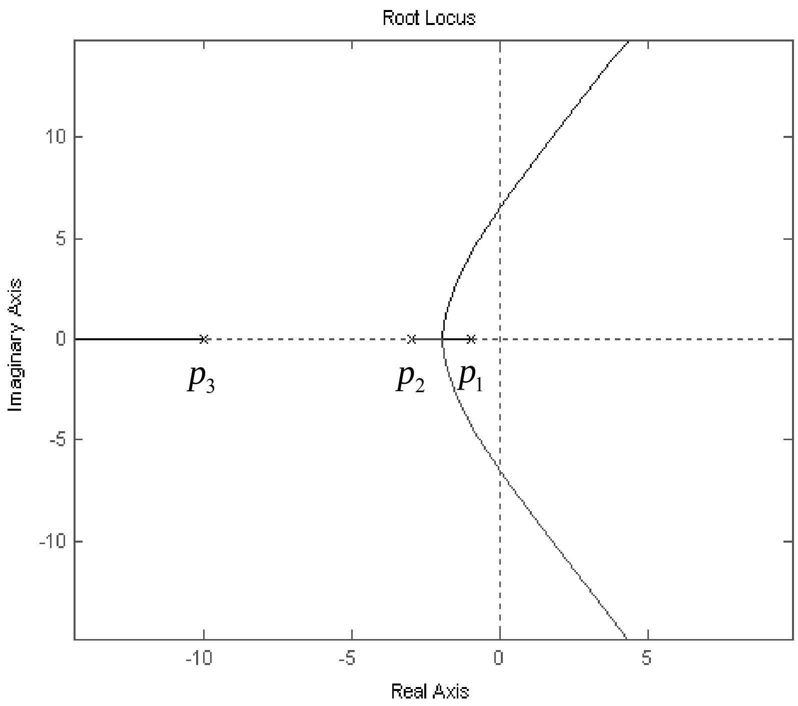


Fig. 4.12.
Root-Locus of an amp with three poles. The arrows show the pole movement as $A_0\beta$ is increased.

Stability and Pole Location

- Note that an amp with three poles has a phase shift of -270° as $\omega \to \infty$.
- So there exists a finite frequency ω_{180} at which the loop gain has 180° phase shift!
- Nyquist plot is another way of looking at the loop stability i.e. plotting $A\beta$ against ω. (Fig. 4.13)
- Note that there is one Nyquist plot for each β. Obviously as β is increased the radial distance $|A\beta|$ is increased so the chance of encirclement of (-1,0) is increased as well.
- For given β: Note for ω=0: $\beta A(s)\Big|_{s=0} = \dfrac{\beta A_0}{(1+s/p_1)(1+s/p_2)(1+s/p_3)}\Bigg|_{s=0} = \beta A_0$

  So $A\beta|_{\omega=0}$ is real, $\angle A\beta$=0 [i.e. Im($A\beta$)=0].
- As ω increases $|A\beta|$ decreases and $\angle A\beta$ becomes negative so plot is in the fourth quadrant.
- As $\omega \to \infty \Rightarrow |A\beta| \to 0$, $\angle A\beta \to -270°$. So the plot is asymptotic to the origin and is tangent to the imaginary axis.
- At ω_{180} frequency, $\angle A\beta$=-180° and the curve crosses the negative real axis.

Stability and Pole Location

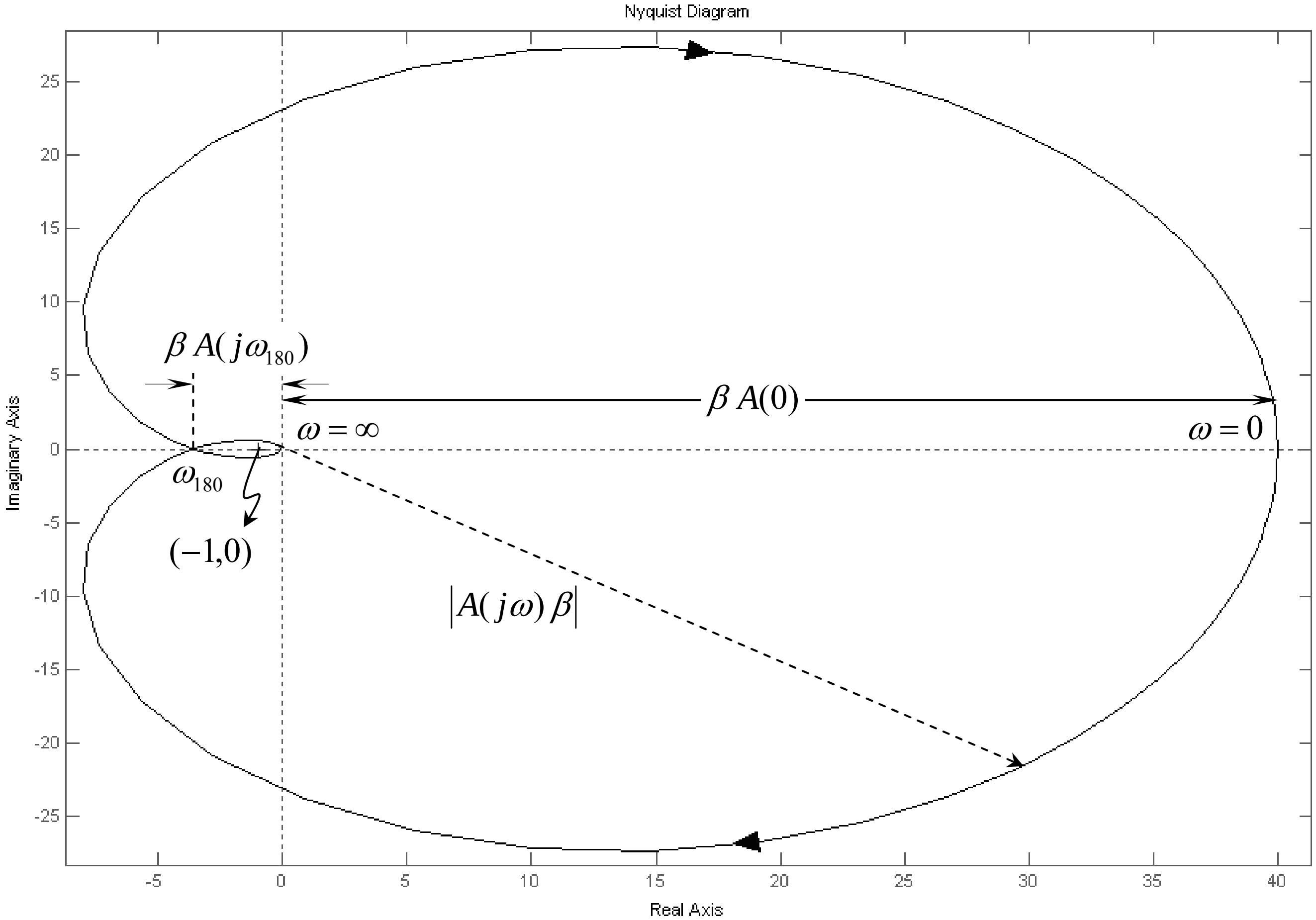


Fig. 4.13.

Stability and Pole Location

- There is a $A_0\beta$ value for which the plot passes through (-1,0) point. Reducing $A_0\beta$ below that value causes Nyquist plot to shrink and so intersect the negative real axis to right of (-1,0) point, indicating stable amp. On the other hand increasing $A_0\beta$ above the critical value expands the plot, thus encircling (-1,0), indicating unstable performance!
- So there is a max β for which the feedback amp becomes unstable.
- Alternatively there is a minimum value for the closed-loop gain A_{f0} below which the amp becomes unstable!

$$\left(A_f = \frac{A(s)}{1+\beta A(s)} \quad \rightarrow \quad \text{higher}\,|\beta A(s)|\,\text{makes}\,|A_f|\,\text{smaller.}\right)$$

- To obtain lower values of closed-loop gain one needs therefore to alter the loop transfer function $L(s)=\beta A(s)$.
- This is the process known as freq. compensation.

Stability and Pole Location

- Note the lower $|A_f| \approx 1/\beta$ (for larger A) means that ω_{-3dB} in amp is higher and φ_M will be lower. This will be seen in φ_M in next section.