# APPROXIMATE ANALYSIS OF NONLINEAR SYSTEMS DRIVEN BY GAUSSIAN WHITE NOISE

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## ABSTRACT

**In this paper an approximate analysis approach for high order nonlinear systems driven by Gaussian white noise is presented. The algorithm is based on the exact solution to the Fokker-Planck equation of a second order system and an optimal model reduction technique. An illustrative example is given to show the application of this approximate approach from which it can be seen that the accuracy of the method is better than that of the random describing function method.**

**Keywords:** Nonlinear systems; Stochastic control; Simulation analysis; Model reduction; Fokker-Planck equation.

## INTRODUCTION

Nonlinear elements are present in all practical control systems, although sometimes they may be replaced by a linearised model without losing too much accuracy in analysis and design. In practical control systems, inputs are not always deterministic, so it is therefore necessary to analyse the statistical behaviour of the system when it is subjected to a random input. To make the analysis procedure easier it is often assumed that the disturbance noise signal is white although such a signal cannot exist in a real system.

Because of their nature, nonlinear systems are more difficult to analyse than their linear counterparts. When stochastic inputs are involved they are often even more difficult to study. There are a few ways to analyse nonlinear systems, one, for instance, is the random describing function approach, also known as the statistical linearisation method, which assumes that the nonlinearity input signal in a nonlinear system is Gaussian when the system is subjected to a Gaussian input signal. This method neglects the distortion caused by the nonlinear element and thus may not be very accurate in some cases. However, to date there is no universal approach available. If the nonlinear system of Figure 1 is driven by a Gaussian white noise, it can be shown that the signals in the system satisfy a Fokker-Planck equation (FPE) [1]. However, FPEs are very difficult, if not impossible, to solve for systems with high order dynamics. In a recent paper [2], exact solutions have been found for first and second order systems with a linear segmented nonlinearity. In this paper, an approximate solution approach for high order FPEs is presented where the optimally reduced models [3], which satisfy certain criteria, replace the original plant model.

An example is given in this paper to show the use of this approximate approach. The accuracy of the approach is compared with simulation results and those obtained using the well known approximate random describing function (RDF) method.

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## EXACT SOLUTION TO SECOND ORDER FOKKER-PLANCK EQUATION

The system structure to be studied is shown in Figure 1. In this system, the signal $r(t) = c$ a constant, and the signal $\gamma(t)$ is Gaussian white noise with mean $\eta$ and a variance of $D$. The

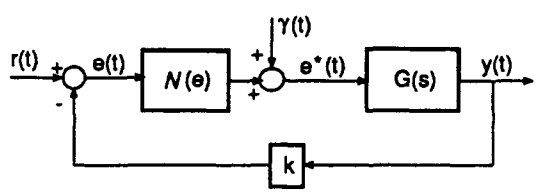


Figure 1: *Typical nonlinear feedback system*

nonlinearity is assumed to be piecewise linear which can be described mathematically as

$$\mathcal{N}(e) = k_i e + b_i, \text{ when } e_i \leq e(t) < e_{i+1}$$

Assuming the nonlinearity has $N$ linear segments, then $e_1 = -\infty$, and $e_{N+1} = \infty$.

If the plant model $G(s)$ is given by a second order model

$$G(s) = \frac{d_2 s + d_1}{s^2 + a_2 s + a_1}$$

the probability density function $p(e)$ of the error signal $e(t)$ can be obtained by solving the corresponding steady-state Fokker-

Planck equation to give the solution [2]

$$p_i(e) = C_1 \prod_{j=1}^{i-1} \delta C_j exp\left[-\frac{\lambda_i(\alpha_i e^2 + 2\beta_i e)}{k^2 D(d_1^2 + \alpha_i d_2^2)}\right] \quad (1)$$

where $\alpha_i = a_1 + kk_i d_1$, $\lambda_i = a_2 + kk_i d_2$, $\beta_i = kd_1(b_i + \eta) - a_1 c$

$$C_1 = \frac{1}{\sum_{i=1}^{N}\left(\prod_{j=1}^{i-1} \delta C_j \Phi_i\right)} \quad (2)$$

and

$$\delta C_j = exp\left\{\frac{1}{k^2 D}\left[\frac{\lambda_{j+1}(\alpha_{j+1} e_{j+1}^2 + 2\beta_{j+1} e_{j+1})}{d_1^2 + \alpha_{j+1} d_2^2} - \frac{\lambda_j(\alpha_j e_{j+1}^2 + 2\beta_j e_{j+1})}{d_1^2 + \alpha_j d_2^2}\right]\right\} \quad (3)$$

The $\Phi_i$'s can be evaluated as follows:

- *If* $\alpha_i \neq 0$, then

$$\Phi_i = \sqrt{2\pi} F_i\left\{PI\left[\frac{e_{i+1} + \beta_i/\alpha_i}{\mu_i}\right] - PI\left[\frac{e_i + \beta_i/\alpha_i}{\mu_i}\right]\right\}$$

where

$$\begin{cases} \mu_i = k\sqrt{\dfrac{D}{2\lambda_i \alpha_i}(d_1^2 + \alpha_i d_2^2)} \\ F_i = \mu_i exp\left[\dfrac{\lambda_i \beta_i^2}{k^2 \alpha_i D(d_1^2 + \alpha_i d_2^2)}\right] \\ PI(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} exp\left(-\frac{1}{2}e^2\right) de \end{cases}$$

- *If* $\alpha_i = 0$, $\beta_i \neq 0$, *then*

$$\Phi_i = \frac{k^2 d_1^2 D}{2\lambda_i \beta_i}\left[exp\left(-\frac{2\lambda_i \beta_i e_i}{k^2 d_1^2 D}\right) - exp\left(-\frac{2\lambda_i \beta_i e_{i+1}}{k^2 d_1^2 D}\right)\right]$$

- *If* $\lambda_i = 0$, *and/or* $\alpha_i = \beta_i = 0$, *then* $\Phi_i = e_{i+1} - e_i$.

## Approximate Solutions to High Order Fokker-Planck Equations

It is required to analyse the system of Figure 1 with a high order plant for which the Fokker-Planck equation cannot be solved. To do this it is proposed to first replace the high order plant by its 'best' reduced second order model. Suppose that the linear plant model $G(s)$ in the system shown in Figure 1 is given by

$$G(s) = \frac{n_1 s^{n-1} + n_2 s^{n-2} + \ldots + n_n}{s^n + q_1 s^{n-1} + \ldots + q_n} \quad (4)$$

and the second order reduced model is given by

$$G_{1/2}(s) = \frac{d_2 s + d_1}{s^2 + a_2 s + a_1}$$

The parameter vector $\theta$ is defined to contain the information of the reduced order model as

$$\theta = (d_1 \quad d_2 \quad a_1 \quad a_2) \quad (5)$$

If there exists a D. C. component in the system, ie., if the nonlinearity is not odd-symmetrical, and/or $d \neq 0$ or $\eta \neq 0$, in order to keep the steady-state value of the system unchanged, it has to be assumed that $d_2/a_1 = n_n/q_n$. Thus in this case the parameter vector $\theta$ can be reduced to

$$\theta = (d_1 \quad a_1 \quad a_2) \quad (6)$$

Several model reduction techniques are available but it is desired to produce a reduced model which is an optimal model for the expected input [3]. An objective function for model reduction can be defined by

$$J = \min_{\theta}\left(\int_0^{\infty} f(t)\epsilon^2(\theta, t)dt\right) \quad (7)$$

where $f(t)$ is a weighting function. The error signal $\epsilon(\theta, t)$ is defined as the difference between the output of the original model $G(s)$ and that of the reduced model $G_{1/2}(s)$ when subjected to the same input signal. The error signal $\epsilon(\theta, t)$ is explicitly written as a function of the parameter vector $\theta$ as well as the time variable $t$.

For linear models $G(s)$ and $G_{1/2}(s)$ and several typical weighting functions $f(t)$, the objective function can be represented in its frequency domain form which corresponds to the minimisation of a function of the mean squared error for a random input [3]

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathcal{H}(s)\mathcal{H}(-s)ds \quad (8)$$

Here $\mathcal{H}(s)$ is a transfer function-like representation which depends upon the parameters of the reduced model as well as the weighting function $f(t)$. For instance, if the weighting function is assumed to be $f(t) = 1$, then the integral squared error (ISE) criterion or mean squared error criterion for a random input is defined from Eq(8) and the function $\mathcal{H}(s)$ equals $E(s)$, where $E(s)$ is the Laplace form of the error signal $\epsilon(t)$ given by

$$E(s) = \left[G(s) - G_{1/2}(s)\right]R(s)$$

and $R(s)$ is the Laplace transform of the input function. If $f(t)$ is given by $f(t) = e^{-\alpha t}$, then the function $\mathcal{H}(s)$ can be written as $\mathcal{H}(s) = E(s + \alpha)$.

If a stable and strictly proper transfer function $\mathcal{H}(s)$ is given, the objective function defined in Eq(8) can be evaluated recursively [4]. Powell's optimisation algorithm [5] can be used for the optimisation process to find an optimal reduced order model. Three kinds of 'optimal' reduced models can be used to replace the original plant model $G(s)$,

1. Assume $e^*(t)$ is Gaussian white noise then find the optimal reduced model using directly the optimal reduction algorithm given in [2]. This requires the mean squared error between the output of $G(s)$ and that of the reduced order model $G_{1/2}^{*1}(s)$ when they are both subjected to the same white noise input to be minimised. In this case

$$\mathcal{H}(s) = G(s) - G_{1/2}^{*1}(s) \quad (9)$$

where $G_{1/2}^{*1}(s)$ is the second order optimally reduced model of the original system $G(s)$.

2. Use the random describing function $K_{eq}(\sigma)$ to replace the nonlinearity, then perform optimal model reduction using the pre-filter

$$G_p(s) = \frac{1}{1 + kK_{eq}(\sigma)G(s)} \tag{10}$$

where $\sigma^2$ is the variance of the error signal $e(t)$. This produces the best estimate for the spectrum of the random input to the nonlinearity. The gain $K_{eq}(\sigma)$ depends upon the parameters of the closed loop system as well as the variance of the input signal $\gamma(t)$. From the original system, the variance $\sigma^2$ of the signal $e(t)$ can easily be found by solving the nonlinear equation

$$\sigma^2 - \frac{D}{2\pi j}\int_{-j\infty}^{j\infty} \mathcal{F}(s)\mathcal{F}(-s)ds = 0 \tag{11}$$

where

$$\mathcal{F}(s) = \frac{kG(s)}{1 + kK_{eq}(\sigma)G(s)}$$

For a particular nonlinear element, an iteration technique can be used to find $K_{eq}(\sigma)$ and the variance $\sigma^2$ of the error signal $e(t)$ [2]. It can easily be seen that $G_p(s)$ is the transfer function from $\gamma(t)$ to $e^*(t)$ and the mean squared error is minimised as in Item 1 but for a coloured noise input similar to that actually existing at the nonlinearity input in the feedback loop. In this case,

$$\mathcal{H}(s) = G_p(s)\left[G(s) - G^{*2}_{1/2}(s)\right] \tag{12}$$

3. Perform the closed loop suboptimal model reduction to the nonlinear system using the algorithm given in [6]. In order to find the approximate solution of the Fokker-Planck equation, it has to be assumed that the deterministic input signal $r(t) = 0$. The diagram to define the error signal for the closed loop reduction algorithm is shown in Figure 2. Here the mean squared error between

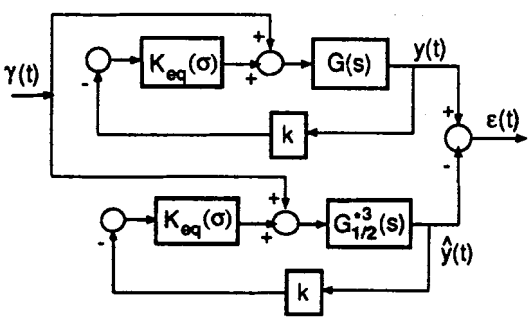


Figure 2: *Block diagram for reduction error*

outputs $\overline{y(t)}$ and $\hat{y}(t)$ is minimised corresponding to the situation where the closed loop system has plant $G(s)$ and then when it is replaced by its reduced order model $G^{*3}_{1/2}(s)$. In this case, the error signal can be written as

$$E(s) = \frac{1}{1 + K_{eq}(\sigma)G(s)} \frac{G(s) - G^{*3}_{1/2}(s)}{1 + K_{eq}(\sigma)G^{*3}_{1/2}(s)} \tag{13}$$

For different $f(t)$, the function $\mathcal{H}(s)$ can be found.

Once the original high order plant model is replaced by a second order one, the exact solution to the reduced order system can be found using the algorithm in Section 2. In the next section, this approximate approach will be compared with another well known approximate method, the RDF method, as well as simulation results.

## An Illustrative Example

Suppose that the plant model $G(s)$ is given by the fourth order transfer function

$$G(s) = \frac{14s^3 + 248s^2 + 900s + 1200}{s^4 + 18s^3 + 102s^2 + 180s + 120}$$

and the nonlinearity is one with dead zone given by

$$\mathcal{N}(e) = \begin{cases} me - m\delta\ sgn(e), & |e| > \delta \\ 0, & |e| \le \delta \end{cases}$$

Also, it is assumed that $c = 0$, $\eta = 0$ and $D = 3$. If we select $m = 1$, and $\delta = 2$, it can be shown that the random describing function $K_{eq}(\sigma) = 0.6969$. The reduced order models using the different methods are

$$G^{*1}(s) = \frac{13.0724s + 9.8986}{s^2 + 2.1513s + 0.9899}$$

$$G^{*2}(s) = \frac{14.6908s + 42.3075}{s^2 + 5.1758s + 4.2307}$$

$$G^{*3}(s) = \frac{14.4693s + 3.3826}{s^2 + 2.0711s + 0.3383}$$

The Nyquist plots for the original plant model, together with those of the reduced models are shown in Figure 3. It can be seen that the high-frequency responses of the reduced models are close to that of the original model.

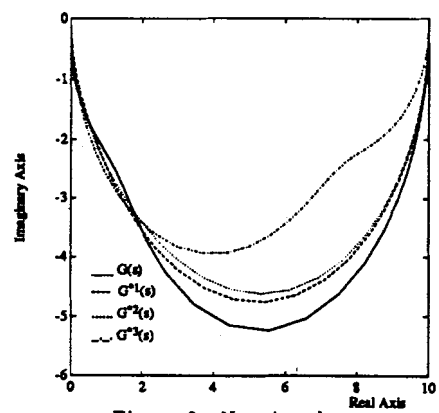


Figure 3: *Nyquist plots*

The approximate Fokker-Planck equation solutions for the different reduced order models are tabulated in Table 1. It can be seen that the mathematical descriptions for the probability density functions of the error signal are nearly the same, although the reduced order models differ significantly.

Table 1: *Approximate solutions to* FPEs

| Model | Fokker-Planck equation solution |
|---|---|
| $G^{*1}(s)$ | $p(e) = \begin{cases} 0.07104exp(-0.00264e^2), & \lvert e\rvert \le 2 \\ 0.0641exp(-0.0282e^2 + 0.1026\lvert e\rvert), & \lvert e\rvert > 2 \end{cases}$ |
| $G^{*2}(s)$ | $p(e) = \begin{cases} 0.06896exp(-0.0027e^2), & \lvert e\rvert \le 2 \\ 0.06265exp(-0.02604e^2 + 0.0947\lvert e\rvert), & \lvert e\rvert > 2 \end{cases}$ |
| $G^{*3}(s)$ | $p(e) = \begin{cases} 0.06891exp(-0.00284e^2), & \lvert e\rvert \le 2 \\ 0.06258exp(-0.02595e^2 + 0.0944\lvert e\rvert), & \lvert e\rvert > 2 \end{cases}$ |

The probability density functions $p(e)$ are drawn in Figure 4, together with the histogram obtained by simulation and the RDF result. It can be seen from Figure 4 that, the results obtained by using $G^{*2}_{1/2}(s)$ and $G^{*3}_{1/2}(s)$ are very close, in fact almost identical so that they cannot be distinguished from the plots, even though the reduced models are different. They give better approximations than using the reduced model $G^{*1}_{1/2}(s)$, which however gives better results than the RDF approach.

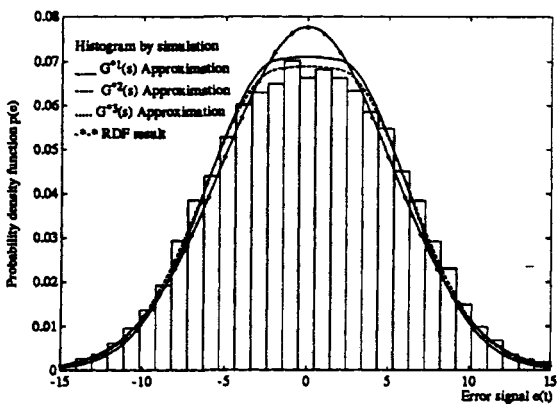


Figure 4: *Probability density function plots*

The approximate solutions of the Fokker-Planck equation are shown in Figure 5, for varying the slope of the nonlinear element whilst keeping $\delta = 2$ and $D = 3$, together with the ones obtained by the RDF approach and the histograms obtained from the simulation results. It can be seen that when the value of the slope $m$ increases, the 'size' of the nonlinearity of the system increases, this makes the accuracy of the RDF method decrease. When $m$ is chosen to be a small value, accuracy of the approximate solution is similar to the one by the RDF method, however, when the value of $m$ increases, it can be seen from the figure that the accuracy of the method given in this paper is better than the RDF approach.

Figure 6 shows the approximate FPE solutions, together with the histograms by simulation and the results of the RDF method for changing the width of the nonlinearity dead zone whilst keeping $m = 1$, and $D = 3$, and Figure 7 shows the ones for varying the variance of the input signal whilst keeping $m = 1$, and $\delta = 2$. It can be seen that in the former case, when the

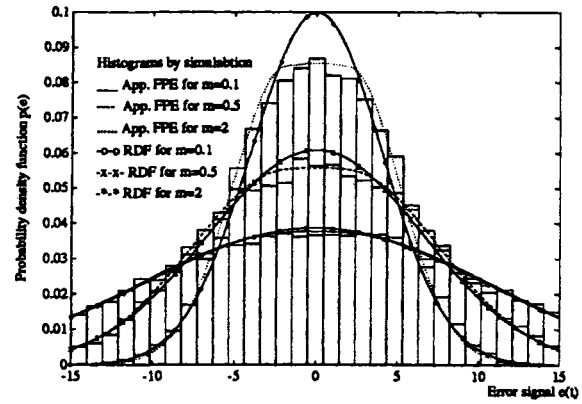


Figure 5: *Approximate solution for different m*

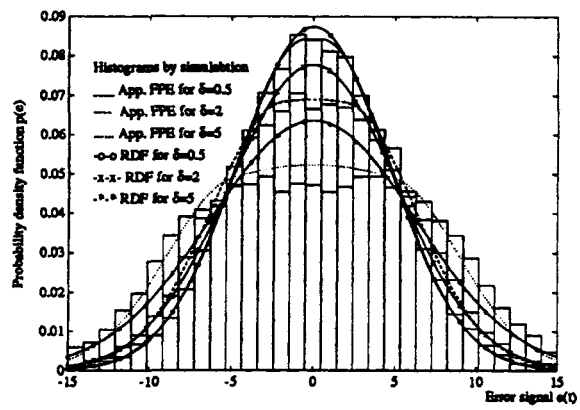


Figure 6: *Approximate solution for different δ*

value of $\delta$ increases, the 'size' of the nonlinearity of the system increases, thus the accuracy of the RDF solution decreases, however, the approximate solution of the corresponding FPEs is still very accurate.

In the latter case, when the variance $D$ of the input signal decreases, the nonlinearity of the system also decreases, and the accuracy of the FPE method is superior to that of the RDF method.

It is obvious from the above results that the accuracy of the RDF depends upon the 'magnitude' of the signal at the input to the dead zone nonlinearity. This will, of course, also be affected by the system transfer function as is demonstrated by considering similar models having different bandwidth. The original plant trandfer function in pole-zero form is

$$G(s) = \frac{14(s + 13.3919)(s + 2.1612 \pm j1.3152)}{(s + 1.1967 \pm j0.6934)(s + 7.8033 \pm j1.3576)}$$

and we consider different models whilst keeping the steady-states the same and changing the two pole positions at $s_{1,2} = -1.1967 \pm j0.6934$ to give the two transfer functions

$$G_1(s) = \frac{36.5953(s + 13.3919)(s + 2.1612 \pm j1.3152)}{(s + 2)(s + 2.5)(s + 7.8033 \pm j1.3576)}$$

and

$$G_2(s) = \frac{2.4395(s + 13.3919)(s + 2.1612 \pm j1.3152)}{(s + 1)(s + 0.3333)(s + 7.8033 \pm j1.3576)}$$

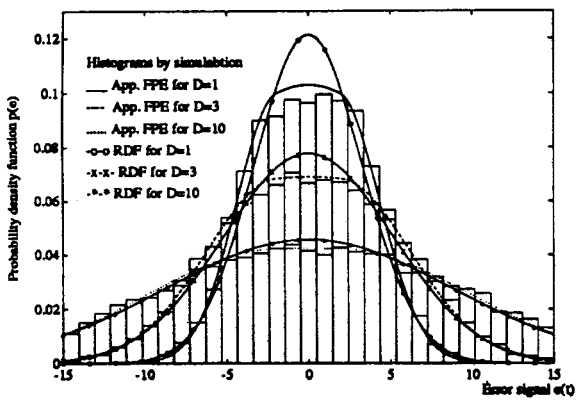


Figure 7: *Approximate solution for different D*

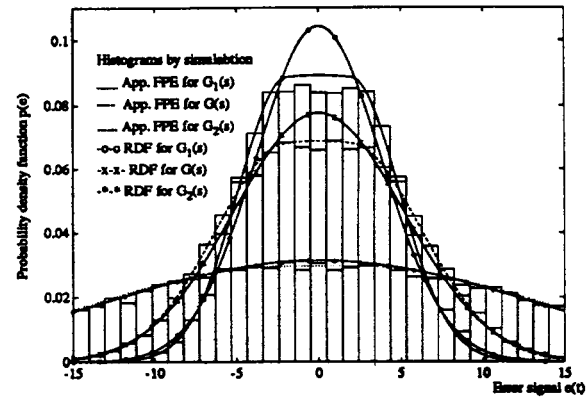


Figure 9: *Approximate solution for different models*

The Bode diagrams of $G(s)$, $G_1(s)$, and $G_2(s)$ are shown in Figure 8. It can be seen that the bandwith of the model $G_1(s)$ is the smallest of the three; and the bandwith of $G_2(s)$ is the largest. This means that the 'size' of the nonlinearity of the system decreases in the order of $G_1(s)$, $G(s)$ and $G_2(s)$. The

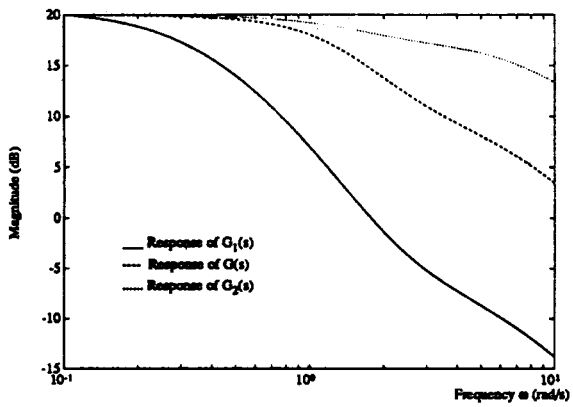


Figure 8: *Bode diagram for different models*

approximate FPE solutions are shown in Figure 9, together with the histograms from the simulation results and the results of the RDF approach. It can be seen that the wider the bandwith of the plant model, the more accurate the RDF approach. The accuracy of the approximate FPE approach is very good, even when the system is highly distorted by the nonlinear element.

It can be concluded from the above analysis that the accuracy of the RDF method depends heavily upon the 'size' of nonlinearity of the system to be studied.

## CONCLUDING REMARKS

In this paper, an approximate approach for the analysis of nonlinear systems driven by Gaussian white noise has been presented. The approach replaces the original plant model by an optimally reduced order one and then finds the exact Fokker-Planck equation solution for the reduced system. An illustrative example has been given which compares the approximate solutions using different reduction criteria with simulation results. The qualitative relationship between the accuracy of the approximation and the parameters of the linear and nonlinear elements and the variance of the input signal have been examined. It is found from the example that the accuracy of the new approach presented in this paper is superior to that of the random describing function method, which is often considered as a good approximate method, when the effect of the nonlinearity in the systems are significant.

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